SHIFT ERROR
IN ITERATED RATIONAL FILTER BANKS

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ABSTRACT

For FIR filters, limit functions generated in iterated rational schemes are not invariant under shift operations, unlike what happens in the dyadic case: this feature prevents an analysis iterated rational filter bank (AIRFB) to behave exactly as a discrete wavelet transform, even though an adequate choice of the generating filter makes it possible to minimize its consequences. This paper indicates how to compute the error between an "average" shifted function and these limit functions, an open problem until now. Also connections are pointed out between this shift error and the selectivity of the AIRFB.

1 INTRODUCTION

AIRFB have been introduced [1] in order to provide a finer "wavelet-like" frequency decomposition than what is available with octave band analysis [2]. The interest of requiring a greater spectral accuracy arises from *e.g.* audio coding: it is known that, for frequencies higher than 500 Hz, the auditory systems performs a (roughly) third-of-octave analysis. The generalization from the dyadic case is rather straightforward and is done by the substitution of the classical "filter—downsampler" branch by the following one

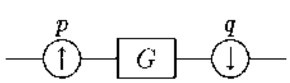

Figure 1: rational branch

where $q > p$. This allows sampling by the fractional factor q/p. On the synthesis side of course, the same kind of branch is used: instead, the condition $q < p$ applies.

It has first been claimed that, the iteration of a rational oversampling branch with FIR filter G could not converge to a limit function [1], as it would otherwise be the case with a dyadic branch. However, we have shown [3] that the situation is slightly more complicated, and that, in fact, there exists limit functions: unlike the dyadic limit functions, these new functions are not shift invariant — which allowed to say that the dyadic schemes converge to *one* function $\varphi(t)$, whereas the correct result is that they converge to *integer shifted versions* $\{\varphi(t-n)\}_{n\in\mathbf{Z}}$ of one function . This difference is indeed the subject of the present paper.

We shall first remind some results, taken mainly from [3], about the limit functions. Then we shall set two different measures for the shift error, and show that one of them — the $\mathbf{L}^2$ definition— is always accessible through an exact computation, and that upper bounds exist for the other one — the $\mathbf{L}^\infty$ one— . Some examples will follow, showing how to choose a filter G such that its shift error be minimized. Finally, it is shown that the $\mathbf{L}^2$ shift error has an unexpected practical interpretation in the analysis filter bank as a selectivity parameter.

All these results form part of a PhD thesis held recently about iterated rational filter banks [4] in which different issues have been addressed, ranging from the description of the limit functions (regularity [5], shift error,...), the properties of a perfect reconstruction rational filter bank (polyphase matrix, statistics, filter design [6],...) to a practical application (audio coding [7]).

2 ITERATED RATIONAL SCHEMES

2.1 Limit functions and time-scale transform

When an oversampled rational branch such as in figure 1 is iterated j times, the impulse response of the system —assuming a dirac impulse at time $t = n$ — tends to behave as $\left\{\varphi_n\left(\frac{q^j}{p^j}k\right)\right\}_{k\in\mathbf{Z}}$, when j tends to infinity. The conditions for this convergence to hold are given in [3], and from now we shall consider that the considered schemes converge strongly. The essential difference with the dyadic case is thus the fact that the index n cannot be substracted from the variable. Otherwise the functions verify a two-scale equation [3]

$$\varphi_n(t) = \sum_k g_{kq-np}\varphi_k\left(\frac{p}{q}t\right) \tag{1}$$

an extension of the two-scale difference equation well known in the dyadic case [8]. This makes it straightforward to generalize the multiresolution theory [2,9] to this case, the multiresolution spaces V_n being generated no more by integer shifted "father" functions but by the functions φ_n: the inclusion property of the V_n is preserved. We could say that the corresponding analysis, still preserving the scale invariance, is the consequence of the use of a measuring device whose characteristics change with time.

As a consequence of the existence of support bounded limit functions, the outputs of an AIRFB can be rewritten as samples of a time-scale transform, with p/q as scale factor [3]. If the shift error were cancelled, then this transform would be a wavelet transform. Finally, minimizing the shift error implies that the AIRFB behaves "almost" as a wavelet transform, which is a strong motivation for the study of this error (this is not the only reason, as we shall see in section 4)

2.2 An average function

Even though the "pseudo-wavelets" have different shapes, there is a limit to those variations and in fact, they look like, more or less, to an "average function"

$$\varphi(t) = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \varphi_n(t+n) \tag{2}$$

which has special scaling properties, derived from a two-scale difference equation. This scaling equation can be simply expressed in the Fourier space by

$$\hat{\varphi}(\nu) = \frac{1}{p} G\left(e^{-2i\pi\frac{\nu}{p}}\right) \hat{\varphi}\left(\frac{q}{p}\nu\right) \tag{3}$$

which shows that the spectrum of this average function can be computed through an infinite product.

3 COMPUTATION OF THE SHIFT ERROR

We shall evaluate the lack of shift invariance of the limit functions in two different ways, through the $\mathbf{L}^\infty$ measure ε

$$\varepsilon = \sup_{n,t} |\varphi_n(t) - \varphi(t-n)| \tag{4}$$

and through the $\mathbf{L}^2$ measure η

$$\eta^2 = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \|\varphi_n(t) - \varphi(t-n)\|_2^2 \tag{5}$$

where $\|.\|_2$ is the natural $\mathbf{L}^2$ norm of square integrable functions.

3.1 Exact computations

It turns out that the $\mathbf{L}^2$ shift error can *always* be exactly computed, whereas this is seldom the case for the $\mathbf{L}^\infty$ one.

3.1.1 The $\mathbf{L}^2$ case

Let us consider the symetrized filter $\Gamma(z) = G(z)G(z^{-1})$. The associated iterated schemes induce limit functions $\Phi_n(t)$ and an average function $\Phi(t)$. Then it can be shown that

$$\eta^2 = \Phi_0(0) - \Phi(0) \tag{6}$$

This expression is easily computed

- $\Phi_n(0)$ is obtained through the resolution of a linear system (thanks to the two-scale equation (1)), but of course, one can simply iterate the schemes to find this value
- $\Phi(0)$ which is also the $\mathbf{L}^2$ norm of $\varphi(t)$ is obtained by considering the Fourier series expansion of $\varphi(t)$. The final result is

  $$\Phi(0) = \frac{1}{\Lambda} \sum_n \left|\hat{\varphi}\left(\frac{n}{\Lambda}\right)\right|^2 \tag{7}$$

  where Λ denotes the size of the support of $\varphi(t)$. The different elements of this sum are computed through the infinite product meant by (3), while the number of terms to be taken into account depends on the convergence rate of the sum, and on the selectivity of $\hat{\varphi}$. Indeed, if we truncate this sum, the result given by (6) will be an upper bound.

Apart from the fact that it allows exact computations, this expression shows that the square $\mathbf{L}^2$ error associated to $G(z)$ is the $\mathbf{L}^\infty$ error associated to $G(z)(G(z^{-1})$.

3.1.2 the $\mathbf{L}^\infty$ case

The preceding computation gives us a clue to obtain ε in some restrictive cases. Assume that $G(z)$ can be written under the form $z^N P(z)P(z^{-1})$ and that $p-q=1$, then we have the following exact value for the $\mathbf{L}^\infty$ shift error

$$\varepsilon = \varphi_{-N}(0) - \varphi(N) \tag{8}$$

in which, as for η, the different ingredients are easy to obtain. We thus see here, that the expression defining (4) reduces to the computation of the shift error at time $t=0$ and for the limit function index $n=-N$.

3.2 Upper bounds

In some cases, it might be necessary to have access only to ε, even if η gives valuable informations. In that case, and if $G(z)$ hasn't the special form given above, it is necessary to find upper bounds for this error. Historically, these bounds have been computed [5] before the exact results given in the preceding subsection. Their interest remains in the fact that they provide a better understanding of the influence of the regularity on the shift error: as a matter of fact, it had been —wrongly— conjectured that the higher the regularity, the lower the shift error.

The idea underlying the computation of the upper bounds, is that ε should not be very different from the error generated in (1) after the replacement of $\varphi_n(t)$ by $\varphi(t-n)$. That is to say, if we define

$$\varepsilon^0 = \sup_{n,t} \left| \varphi(t-n) - \sum_k g_{kq-np}\varphi\left(\frac{p}{q}t - n\right) \right|$$

then ε^0 might be a good approximation of ε. Notice that the quantity ε^0 is easy to compute, either by Fourier transform since

$$\varepsilon^0 \leq \frac{q-1}{q} \max_{k=1}^{q-1} \int |\hat{\varphi}(\nu)| \left| G\left(e^{-2i\pi\frac{\nu+k}{q}}\right) \right| d\nu \qquad (9)$$

(it is important to notice that the "max" does not include the value $k=0$) or in the time domain, since it is sufficient to consider that the index n runs only over q consecutive values.

The obtained upper bounds are rather complicated since it is necessary to define a certain number of different parameters and this would be too long for this article. However it is enough to remember that we can reach formulæ of the form

$$\varepsilon \leq \varepsilon^0 + \rho \qquad (10)$$

where ρ depends, in a complex manner on the regularity and the support of the limit functions. The only thing that is worth mentioning is that ρ decreases as the regularity increases.

4 LINKS SELECTIVITY/SHIFT ERROR

We shall define the selectivity $\sigma(\nu_0)$ of an AIRFB as a function of the frequency ν_0: it is the square root of the upper limit value (when the number of iterations increase) of the energy in the attenuation band relative to the energy in the passband, for the low-pass branch. After j iterations, the low-pass branch is constituted of an up-sampler q^j a filter $G_j(z) = G(z^{p^{j-1}})G(z^{qp^{j-2}})\ldots G(z^{q^{j-1}})$ and a down-sampler p^j. So by definition

$$\sigma^2(\nu_0) = \limsup_{j\to\infty} \frac{\displaystyle\int_{\frac{\nu_0}{2p^j}}^{1/2} \left|G_j(e^{-2i\pi\nu})\right|^2 d\nu}{\displaystyle\int_0^{\frac{\nu_0}{2p^j}} \left|G_j(e^{-2i\pi\nu})\right|^2 d\nu} \qquad (11)$$

The important following result makes the link between this function $\sigma(\nu_0)$ and some properties of the limit functions

$$\sigma^2(\nu_0) = \frac{\eta^2 + \displaystyle\int_{|\nu|\geq\frac{|\nu_0|}{2}} |\hat{\varphi}(\nu)|^2 d\nu}{\displaystyle\int_{|\nu|\leq\frac{|\nu_0|}{2}} |\hat{\varphi}(\nu)|^2 d\nu} \qquad (12)$$

In particular, we can derive the following lower bound for the selectivity

$$\sigma(\nu_0) \geq \frac{\eta}{||\varphi||_2} \qquad (13)$$

which shows that the $\mathbf{L}^2$ shift error can be re-interpreted as a minimal selectivity for the iterated schemes (in general $||\varphi||_2 \approx 1$, especially if we are interested in orthonormal AIRFB).

If we remember the Fourier expression (9) of ε^0, we can understand that there is no surprise in such a link between shift error and selectivity: if we assume $\hat{\varphi}$ to be in $\mathbf{L}^1$ and $G(e^{-2i\pi\nu})$ to be very selective within the frequency interval $[-\frac{1}{p+q}, \frac{1}{p+q}]$ then the value of ε^0 gets very small, which indicates that ε itself can become very small too. The advantage of the result given in (12) is that this link is quantitative.

The fact that selectivity plus some amount of regularity tends to minimize the shift error makes it easier to understand why it was first claimed that regularity was a direct factor influencing this parameter: as a matter of fact, the regularity factor $\frac{z^p-1}{z-1}$ acts as a low-pass filter which thus tends to diminish the frequency support of the filter. But regularity is a much more complex thing than simply low-pass filtering, which explains why it is possible to find counter examples to the conjecture proposed in [3].

5 EXAMPLES

We give here some examples, all based on the exact computation of η and which show how the shift error behaves when some parameters vary.

5.1 Regularity

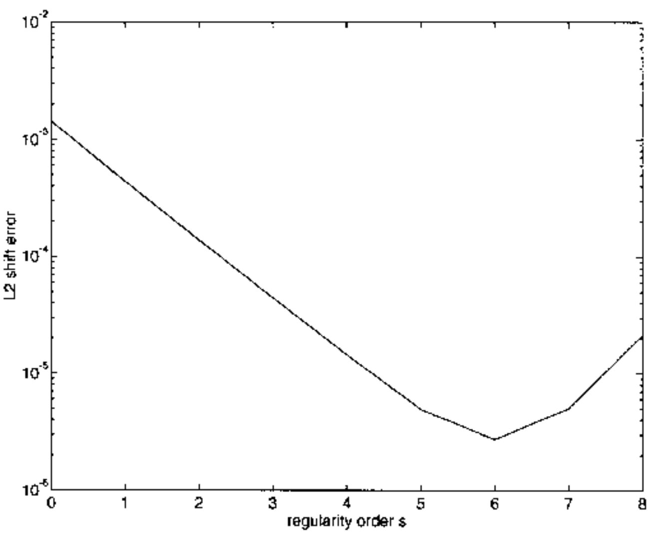

Figure 2: η as a function of s

We shall consider the following generating filters

$$G^s(z) = 3\left(\frac{z^2+z+1}{3}\right)^s \left(\frac{z+1}{2}\right)^{9-s}$$

for $s = 1\ldots 9$ and with $p/q = 3/2$. If we denote by φ_n^s the limit functions generated by G^s, it is known that $\partial\varphi_n^s = \varphi_n^{s-1} - \varphi_{n+1}^{s-1}$. Thanks to this differentiation relation, the regularity order of the functions φ_n^s is exactly of the form $r_0 + s$ where $r_0 + 1$ is the regularity order of the set of functions φ_n^1. The $\mathbf{L}^2$ shift error for different values of s is plotted in figure 2. As we can see in this example, the increase in regularity corresponds to an exponential decrease in η for a wide range of s. However, for the most regular functions, the shift error increases again, pointing out that the link regularity/η is not direct.

5.2 attenuation

We are interested here in the influence of the attenuation on the shift error. For this, we shall use orthonormal filters obtained from [6]. We are still in the 3/2 case and consider filters of length 21 minimizing (or trying to...) an $\mathbf{L}^2$ attenuation for various values of the beginning of the attenuation band. The results are plotted in figure 3 As we can see, the attenuation alone cannot account for

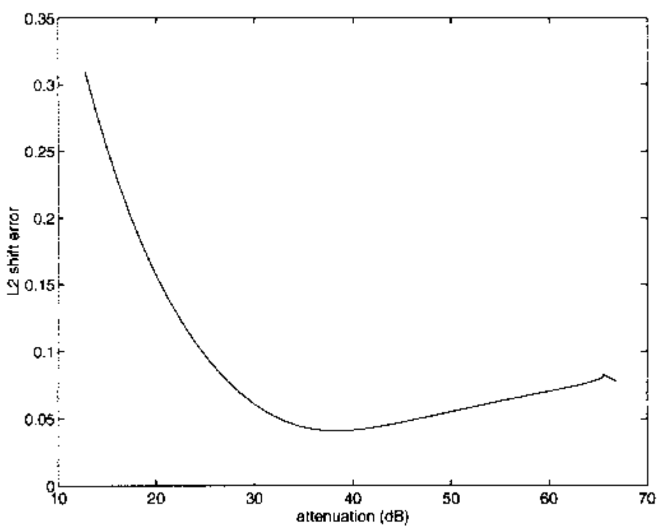

Figure 3: η as a function of the attenuation

the decrease of the shift error: there is a subtle tradeoff between the width of the transition band (directly increasing here with the attenuation) and the attenuation, for which the shift error is minimized.

6 CONCLUSION

In this paper, we have shown the interest of the so-called "shift error" in iterated rational schemes from the AIRFB as well as from the multiresolution —i.e. the limit functions— point of view. We have been able to give exact values for one of the possible measures of this lack of shift invariance, the $\mathbf{L}^2$ shift error. For the $\mathbf{L}^\infty$ shift error, we have only sketched the results given in [4] due to their formal complexity: The author is presently preparing a journal paper fully exposing these results [10].

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