The Optimum Approximation in Generalized Time-Frequency Domains and Application to Numerical Simulation of Partial Differential Equations

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SUMMARY Extended optimum interpolatory approximation is presented for a certain set of signals having n variables. As the generalized spectrum of a signal, we consider a $\nu-$dimensional vector. These variables can be contained in one of the time domain, the frequency domain or the time-frequency domain. Sometimes, these can be contained in the space-variable domain or in the space-frequency variable domain. To construct the theory across these domains, we assume that the number of variables for a signal and its generalized spectrum are different, in general.

Under natural assumption that those generalized spectrums have weighted norms smaller than a given positive number, we prove that the presented approximation has the minimum measure of approximation error among all the linear and the nonlinear approximations using the same generalized sample values. Application to numerical simulation of partial differential equations is considered. In this application, a property for discrete orthogonality associated with the presented approximation plays an essential part.

1. OPTIMUM APPROXIMATION

We denote by R^n and Z^n the set of all the real $n-$dimensional vectors and the set of all the $n-$ dimensional integer vectors, respectively. Let $X=(x_1,x_2,\ldots,x_n)$ and $U=(u_1,u_2,\ldots,u_\mu)$ be real vectors in R^n and R^μ, respectively. Sometimes, the simplified expression $n-D$ vectors is used for the terms n-dimensional one.

Let $P=(p_1,p_2,\ldots,p_n)$ be an integer vector in a subset Δ in Z^n. We denote by $\Lambda(M)$ the set of all the vectors $J=(j_1,j_2,\ldots,j_n)$ in Z^n the elements of which satisfy $1\le j_k\le m_k$ (m_k =positive integer, $k=1,2,\ldots,n$; $M=(m_1,m_2,\ldots,m_n)$).

Suppose that Λ is a subset of $\Lambda(M)$. In the following, we assume that $K=(k_1,k_2,\ldots,k_n)$ is a vector in Λ.

Also, we use the following notations.

$\sum_P=\sum_{p_1}\sum_{p_2}\cdots\sum_{p_n}\quad(P=(p_1,p_2,\ldots,p_n)\in\Delta)$.

$\sum_K=\sum_{k_1}\sum_{k_2}\cdots\sum_{k_n}\quad(K=(k_1,k_2,\ldots,k_n)\in\Lambda)$.

$\Xi=$[a Hilbert space of $\nu-D$ vectors $F(U)=(F^1(U),F^2(U),\ldots,F^\nu(U))=(F^1(u_1,u_2,\ldots,u_\mu),F^2(u_1,u_2,\ldots,u_\mu),\ldots,F^\nu(u_1,u_2,\ldots,u_\mu))$].

$\| F \|=$[a norm of $F(U)$ in Ξ]. For example, $\| F \|=\sqrt{\sum_{q=1}^{\nu}\int_{R^\mu}\mid F^q(U)\mid^2 dU}$. Note that $\| F \|$ is a positive number, and not a vector nor a function of $u_1,u_2,\ldots,u_\mu$.

$(F,G)=$[inner product between $F(U)$ and $G(U)$, where $F(U)$ and $G(U)$ are in Ξ]. For example, $(F,G)=\sum_{q=1}^{\nu}\int_{R^\mu}F^q(U)\overline{G^q(U)}dU$. Here, (F,G) is a complex constant, and not a vector nor a function of $u_1,u_2,\ldots,u_\mu$.

$V=$[an operator matrix on Ξ. Its range is also Ξ].

$V^{-1}=$[an inverse operator matrix of V whose domain and range are Ξ].

$B=$[subset of Ξ composed of $F(U)$ satisfying $\| V^{-1}F \|^2\le A$, where A is the prescribed positive number].

$H_K=$[operator matrices on Ξ having the range in Ξ, where $K=(k_1,k_2,\ldots,k_n)\in\Lambda$].

We assume that all the operators are bounded and linear in their domain.

Now, we consider a series of $\nu-D$ vectors $s(U,X)=(s^1(U,X),s^2(U,X),\ldots,s^\nu(U,X))$ and $s_{K,P}(U)=(s^1_{K,P}(U),s^2_{K,P}(U),\ldots,s^\nu_{K,P}(U))$ $(K\in\Lambda,P\in\Delta)$ in Ξ, where X is a parameter vector.

Then, the signals with n variables treated in this discussion are defined by $f(X)=(V^{-1}F,s(U,X))$ $(F\in B)$. Let Γ be the set of these signals $f(X)$.

For example, let $\mu=n$. Suppose that $V=\mid W(U)\mid^2$ and $V^{-1}=1/\mid W(U)\mid^2$ hold. Also, let $(F,G)=\int\mid W(U)\mid^2 F(U)\overline{G(U)}dU/(2\pi)^n$ and $s(U,X)=exp(-jUX^t)$. Then, above definition of $f(X)$ is identical to ordinary inverse Fourier transform with n variables.

Further, we define (generalized) sample values of $f(X)$ by $f_{K,P}=(V^{-1}H_KF,s_{K,P}(U))$ $(K\in\Lambda,P\in\Delta)$.

The approximation formula is defined by

$$y(X)=\Sigma_K\ \Sigma_P\ f_{K,P}w_{K,P}(X) \tag{1}$$

For convenience sake, we call $w_{K,P}(X)$ $(K\in\Lambda,P\in\Delta)$

(generalized) interpolation functions. The approximation error between $f(X)$ and $y(X)$ is given by $e(X) = | f(X) - y(X) |$. Then, the measure of error $E(X)$ is defined by

$$E(X) = \sup_{f(X)\in\Gamma} [e(X)] \tag{2}$$

which means the upper limit of $e(X)$ for all the $f(X)$ in Γ.

PROBLEM: Assume that V, Γ, H_K, $s(U, X)$ and $s_{K,P}(U)$ $(K \in \Lambda, P \in \Delta)$ are given. Then, derive the optimum $w_{K,P}(X)$ minimizing $E(X)$.

Note that there exists an operator matrix G_K on Ξ satisfying $(V^{-1}H_K V \cdot V^{-1}F, s_{K,P}(U)) = (V^{-1}F, G_K s_{K,P}(U))$ as a consequence of Riesz theorem.

Now, let X be a fixed parameter vector in R^n. Then, $e(X) = | f(X) - y(X) |$ can be expressed as

$$e(X) = | (V^{-1}F, S(U, X)) | \tag{3}$$

where $S(U, X) = s(U, X) - \sum_K \sum_P \overline{w_{K,P}(X)} G_K s_{K,P}(U)$.

Using Schwarz inequality, we obtain $e(X) \leq \| V^{-1}F \| \bullet \| S(U, X) \| \leq \sqrt{A} \| S(U, X) \|$. Equality holds when $V^{-1}F_1(U) = c \bullet S(U, X)$ is valid, where $c = \sqrt{A} / \| S(U, X) \|$, $F_1(U) = F_1(U, X) \in B$ and X is a parameter vector. Note that, if $F(U) = F_1(U)$, then $e(X) = E(X)$ holds. Hence, we can easily prove

$$E(X) = \sqrt{A} \| S(U, X) \| \tag{4}$$

Differentiating $E(X)^2$ with respect to $w_{M,Q}(X)$ $(M \in \Lambda, Q \in \Delta)$ and putting the resultant formulas into zero, we obtain a set of linear equations for the optimum interpolation functions $w_{K,P}(X)$ $(K \in \Lambda, P \in \Delta)$ minimizing $E(X)$.

$$\begin{aligned} &\Sigma_K \Sigma_P w_{K,P}(X) \cdot (G_M(U) s_{M,Q}(U) G_K(U) s_{K,P}(U)) \\ &= (G_M(U) s_{M,Q}(U), s(U, X)) \ (M \in \Lambda, Q \in \Delta) \end{aligned} \tag{5}$$

We assume that the coefficient matrix has sufficient rank.

2. GENERALIZED DISCRETE ORTHOGONALITY

Let $p = d(K, P)$ $(p = 1, 2, \ldots, m; K \in \Lambda, P \in \Delta)$ be one to one correspondence between an integer p $(p = 1, 2, \ldots, m)$ and a pair of vectors (K, P) $(K \in \Lambda, P \in \Delta)$. Further, we assume that the coefficient matrix of Eqs.(5) has sufficiently large rank. Then, as is easily proved, we may consider that the vectors $G_K \cdot s_{K,P}(U)$ $(K \in \Lambda, P \in \Delta)$ are independent with each other. Hence, using the Schmidt orthogonality algorithm, we can derive a set of orthogonal base-vectors $\{v_p(U)\}$ $(p = 1, 2, \ldots, m)$ with respect to the present inner product from the set of the vectors $\{G_K \cdot s_{K,P}(U)\}$ $(K \in \Lambda, P \in \Delta)$:

$$v_p(U) = \sum_{q=1}^{m} a_{p,q} G_M \cdot s_{M,Q}(U)$$

$$G_K \cdot s_{K,P}(U) = \sum_{q=1}^{m} b_{p,q} \cdot v_q(U) \tag{6}$$

where $\| v_p(U) \| = 1$, $(v_p(U), v_q(U)) = 0$ $(p \neq q)$ and $p = d(K, P)$ $(p = 1, 2, \ldots, m; K \in \Lambda, P \in \Delta)$ and $q = d(M, Q)$ $(q = 1, 2, \ldots, m; M \in \Lambda, Q \in \Delta)$. Further, $a_{p,q}$ and $b_{p,q}$ $(p,\ q = 1, 2, \ldots, m)$ are the complex coefficients associated with the Schmidt orthogonality algorithm.

Then, we can obtain

$$\begin{aligned} E(X)^2/A &= \sum_{p=1}^{m} | r_p(X) - (s(U, X), v_p(U)) |^2 \\ &+ (s(U, X), s(U, X)) - \sum_{p=1}^{m} | (s(U, X), v_p(U)) |^2 \end{aligned} \tag{7}$$

where $r_p(X) = \sum_{q=1}^{m} b_{q,p} \overline{w_{M,Q}(X)}$. Further, $p = d(K, P)$ $(p = 1, 2, \ldots, m; K \in \Lambda, P \in \Delta)$, $q = d(M, Q)$ $(q = 1, 2, \ldots, m; M \in \Lambda, Q \in \Delta)$.

Therefore, we may soon notice that the following $r_p(X)$ minimize $E(X)^2$, where $p = 1, 2, \ldots, m$.

$$r_p(X) = (s(U, X), v_p(U)),\ (p = 1, 2, \ldots, m) \tag{8}$$

In the following in this section, we assume that the interpolation functions are identical with the optimum those minimizing $E(X)$ under consideration. Hence, we use the $r_p(X)$ given Eq.(8).

From Eq.(7), we can easily obtain

$$\begin{aligned} w_{K,P}(X) &= \sum_{q=1}^{m} \overline{a_{q,p}} \cdot \overline{(s(U, X), v_q(U))} \\ &= (V_{-1} \cdot W_{K,P}(U), s(U, X)) \end{aligned} \tag{9}$$

$$where\ \ W_{K,P}(U) = V \cdot \sum_{q=1}^{m} \overline{a_{q,p}} \cdot v_q(U) \tag{10}$$

Let L be a vector in Λ and let R be a vector in Δ. Then, from Eq.(6) and Eq.(9), we can prove

$$\begin{aligned} & (V^{-1}H_L \cdot W_{K,P}(U), s_{L,R}(U)) \\ = \ & (V^{-1}H_L V \cdot \sum_{q=1}^{m} \overline{a_{q,p}}\ v_q(U), s_{L,R}(U)) \\ = \ & (\sum_{q=1}^{m} \overline{a_{q,p}}\ v_q(U), G_L \cdot s_{L,R}(U)) \\ = \ & (\sum_{q=1}^{m} \overline{a_{q,p}}\ v_q(U), \sum_{r=1}^{m} \overline{b_{l,r}}\ v_r(U)) = \sum_{q=1}^{m} \overline{b_{l,q} \cdot a_{q,p}} \\ = \ & \begin{cases} 1 & (l = p,\ that\ is,\ (K, P) = (L, R)) \\ 0 & (l \neq p,\ that\ is,\ (K, P) \neq (L, R)) \end{cases} \end{aligned} \tag{11}$$

Eq.(11) shows that the generalized sample values of $w_{K,P}(X)$ satisfy the generalized discrete orthogonality. As a direct consequence of Eq.(11), if $f(X)$ is equal to a linear combination of $w_{K,P}(X)$, the corresponding $y(X)$ is equal to $f(X)$, and $E(X)$ is equal to zero.

3. GENERALIZED SPECTRUM OF y(X)

Firstly, consider the following function of the vector U.

$$Y(U) = V \cdot \sum_{q=1}^{m} (V^{-1}F, v_q(U))v_q(U) \tag{12}$$

where $v_q(U)$ $(q = 1, 2, \ldots, m)$ are the orthonormal bases defined previously. Then, as direct consequences of Eq.(6), Eqs.(9), (10) and (11), we can prove

$$\begin{aligned} y(t) &= \sum_{p=1}^{m} f_{K,P}\ w_{K,P}(X) \\ &= (V^{-1}Y(U), s(U, X)) \end{aligned} \tag{13}$$

Eqs.(13) shows that $Y(U)$ can be considered as the generalized spectrum of the approximation formula $y(X)$.

Further, we define

$$F(U) = Y(U) + \varepsilon(U) \tag{14}$$

Obviously, $\varepsilon(U)$ is the generalized spectrum of the net value of the approximation error $\epsilon(X) = f(X) - y(X)$. Then, we can derive the relation with respect to the generalized sample values and the squared values of the weighted norms (the weighted energy) of $F(U)$, $Y(U)$ and $\varepsilon(U)$.

$$\begin{aligned} & f_{K,P} = (V^{-1}H_K \cdot F(U), s_{K,P}(U)) = \\ & (V^{-1}H_K \cdot Y(U), s_{K,P}(U)) \\ & (V^{-1}H_K \cdot \varepsilon(U), s_{K,P}(U)) = 0 \\ & \| V^{-1}F(U) \|^2 = \| V^{-1}Y(U) \|^2 + \| V^{-1}\varepsilon(U) \|^2 \end{aligned} \tag{15}$$

As a direct consequence of Eq.(15), we can prove that, if $F(U)$ is in B, the corresponding $\varepsilon(U)$ is contained in B.

Now, let Γ_1 be the set of signals $f(X)$ in Γ satisfying that

(a) the corresponding generalized sample values $[\ f_{K,P}\ ]_0 = (V^{-1}H_K \cdot F(U), s_{K,P}(U))$ are all zero with respect to all the K $(\in \Lambda)$ and P $(\in \Delta)$,

(b) the inequality $\| V^{-1}F(U) \|^2 \leq A$, where A is the prescribed positive number.

Further, suppose that $\hat{y}(X) = v[\{y_{K,P}\}; X]$ is a linear or nonlinear approximation with a parameter X $(\in R^n)$ for $f(X)$ in Γ using the sample values $f_{K,P} = (V^{-1}H_K F, s_{K,P}(U))$ $(K \in \Lambda, P \in \Delta)$. We assume that $v[\{y_{K,P}\}; X]$ is zero when all the $f_{K,P}$ $(K \in \Lambda, P \in \Delta)$ are zero. Since the error $\hat{\epsilon}(X) = f(X) - \hat{y}(X)$ depends on the signal $f(X)$, we express the error as $\hat{\epsilon}(X) = \widehat{\xi}[f(X)]$.

Moreover, let $d(X) = \gamma[\hat{\epsilon}(X); X]$ be *an arbitrary kind of linear/nonlinear approximation error* between the signal $f(X)$ in Γ and the corresponding approximation formula $\hat{y}(X)$, where X $(\in R^n)$ is the prescribed parameter vector. Besides, we assume that

$$\sup_{f(X)\in\Theta_1} \gamma[\hat{\epsilon}(X); X] \leq \sup_{f(X)\in\Theta_2} \gamma[\hat{\epsilon}(X); X] \tag{16}$$

holds for all the set of signals Θ_1 and Θ_2 satisfying $\Theta_1 \subseteq \Theta_2$.

The measure of error in this discussion is defined as

$$E(X) = \sup_{f(X)\in\Gamma} \gamma[\hat{\epsilon}(X); X] \tag{17}$$

$d(X)$ may be a function of the prescribed vector X and does not necessarily satisfy the axiom of the distance.

Let $\epsilon(X) = f(X) - y(X)$, where $y(X)$ is the proposed optimum approximation for $f(X)$. Further, let Γ_0 be the set of all the $\epsilon(X) = f(X) - y(X)$ $(f(X) \in \Gamma)$. Then, as the direct consequence of Eq.(15), we can easily recognize that $\epsilon(X) = f(X) - y(X) \in \Gamma_0 \subseteq \Gamma_1$ holds. Further, if $f(X)$ is equal to $\epsilon(X)$, the corresponding approximation formula $y(X) = y(X)_0$ is identical with zero. Hence, we can easily recognize that the following three conditions hold.

(c) $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma$

(d) $\epsilon(X) = \widehat{\xi}[\epsilon(X)]$

(e) $y(X) = 0$ if all $f_K(X_{K,P})$ $(K \in \Lambda, P \in \Delta)$ are zero.

Therefore, for arbitrary $f(X)$ in Γ, we have

$$\begin{aligned} & E(X) = \sup_{f(X)\in\Gamma} \{\gamma[\hat{\epsilon}(X), X]\} \\ & \geq \sup_{f(X)\in\Gamma_1} \{\gamma[\hat{\epsilon}(X), X]\} = \sup_{f(X)\in\Gamma_1} \{\gamma[f(X), X]\} \\ & E_0(X) = \sup_{f(X)\in\Gamma} \{\gamma[\epsilon(X), X]\} \\ & = \sup_{\epsilon(X)\in\Gamma_0} \{\gamma[\epsilon(X), X]\} = \sup_{\epsilon(X)\in\Gamma_0} \left\{\gamma\left[\widehat{\xi}[\epsilon(X)], X\right]\right\} \\ & = \sup_{f(X)\in\Gamma_0} \left\{\gamma\left[\widehat{\xi}[f(X)], X\right]\right\} = \sup_{f(X)\in\Gamma_0} \{\gamma[\hat{\epsilon}(X), X]\} \\ & \leq \sup_{f(X)\in\Gamma_1} \{\gamma[\hat{\epsilon}(X), X]\} = \sup_{f(X)\in\Gamma_1} \{\gamma[f(X), X]\} \end{aligned} \tag{18}$$

As shown in Eq.(18), $E_0(X)$ is the minimum value of $E(X) = \sup_{f(X)\in\Gamma} \gamma[\hat{\epsilon}(X); X]$. Hence, we can conclude that *the presented optimum interpolation functions minimizes various $E(X)$ at the same time.*

4. SOME APPLICATIONS

Now, we apply these discussions to the finite orthogonal series. Consider a signal $f(X) = \sum_{P\in\Lambda(M)} a_P\ \theta_P(X)$, where $a_P = \int_\Upsilon f(Y)\overline{\theta_P(Y)}dY$ $(P \in \Lambda(M))$. Also, we define $v(X, Y) = \sum_{P\in\Lambda(M)} \sum_{Q\in\Lambda(M)} v_{P,Q}\theta_P(X)\overline{\theta_Q(Y)}$, where $\hat{V} = \{v_{P,Q}\}$ is a positive definite Hermitian matrix. Let $v^{-1}(X, Y) = \sum_{P\in\Lambda(M)} \sum_{Q\in\Lambda(M)} v^{-1}_{P,Q}\theta_P(X)\overline{\theta_Q(Y)}$, where $\hat{V}^{-1} = \{v^{-1}_{P,Q}\}$ is the inverse matrix of V, and let V and V^{-1} be the operators $Vf = [Vf](X) = \int_\Upsilon v(X, Y)$ $f(Y)dY$ and $V^{-1}f = [V^{-1}f](X) = \int_\Upsilon v^{-1}(X, Y)f(Y)dY$.

We consider the bilinear form and the generalized norm, $< f, g >= \int_\Upsilon \int_\Upsilon v(X, Y)f(Y)\overline{g(X)}dY dX$ and $\| f \| = < f, f >^{1/2}$, respectively. Moreover, $s(Y, X)$ is defined by $s(Y, X) = \sum_{P\in\Lambda(M)} \overline{\theta_P(X)}\theta_P(Y)$, where X is a parameter vector. Then, we obtain $f(X) =< [V^{-1}f](Y), s(Y, X) >=$ $< V^{-1}f, s(Y, X) >$. Further, let $f_K(X) = [H_K f](X) =$ $\sum_{P\in\Lambda(M)} \left(\sum_{Q\in\Lambda(M)} h^K_{P,Q} a_Q\right) \theta_P(X)$ $(K \in \Lambda)$, where $\hat{H}_K$ $= \{h^K_{P,Q}\}$ is a complex matrices.

Now, we consider the sample points in R^n expressed as X_K $(K = (k_1, k_2, \ldots, k_n) \in \Lambda)$. The sample values are defined as $f_K(X_K)$ $(K \in \Lambda)$. Further, we define $s_K(Y)$ by $s_K(Y) = s(Y, X_K) = \sum_{P\in\Lambda(M)} \overline{\theta_P(X_K)}\theta_P(Y)$. Then, we can obtain $f_K(X_K) =< [V^{-1}H_K f](Y), s_K(Y) >=< V^{-1}$ $H_K f, s_K(Y) >$. Hence, the same analysis is possible.

As the bases of expansion, if an arbitrary finite or infinite set of independent piece-wise polynomials, including biorthogonal spline wavelets, we can construct the set of orthogonal system from these functions by the Schmidt's orthogonalization. This linear coordinate transformation can be included in the transformation matrix $\hat{V}^{-1}$. Besides, if the set of shifted piece-wise polynomials, such as spline wavelets, are used for the bases and continuous partial differentiation of a signal is replaced by discrete difference with respect to the unit delay, this process to the signal can be included in $[H_K f](X)$. Hence, the presented formulation can be applied and *under the previous assumptions, the result gives the minimum approximation error among all the other approximations.*

Secondly, we will consider the analysis for continuous signals. Suppose that Y^t is the transpose of a vector or a matrix Y and $\overline{Y}$ is the conjugate of Y. Let $X * Y$ be a vector $(x_1y_1, x_2y_2, \ldots, x_ny_n)$, where $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$. Further, let $f(X) = f(x_1, x_2, \ldots, x_n)$ and $\int_{\hat{\Theta}} f(X)dX = \int\int\ldots\int f(x_1, x_2, \ldots, x_n)dx_1dx_2\ldots dx_n$ defined for all the X in $\hat{\Theta}$. When a pair of functions $f(X) \rightharpoonup$ $\rightharpoondown F(U)$ satisfies $f(X) = \frac{1}{(2\pi)^n}\int_{R^n} F(U)\exp(jUX^t)dU$, then $F(U)$ is called the Fourier spectrum of $f(X)$.

Let $T = (t_1, t_2, \ldots, t_n)$ and $A_K = A_{k_1,k_2,\ldots,k_n}$ be real constant vectors, respectively. $A_K = A_{k_1,k_2,\ldots,k_n}$ has integer subscripts k_m $(m = 1\text{`}n)$. We assume $t_k > 0$ $(p = 1, 2, \ldots, n)$ and $K \in \Lambda$, where Λ be the prescribed finite set in Z^n. Then, the sample points are defined by $X_{K,P} = (P{*}T{+}A_K)G^t$ $(K \in \Lambda, P \in Z^n)$, where $G^t = H^{-1}$ is a real non-singular matrix for the coordinate transformation. The space R^n is divided into disjoint union of the polyhedra Γ_i $(i = 0, \pm1, \pm2, \ldots)$ each of which is the fundamental period of $X_{K,P}$.

Let $V =| W(U) |^2$ and Ξ be the Hilbert space having the inner product and the norm such as $(F, G) = \frac{1}{(2\pi)^n}\int_{R^n}$ $| W(U) |^2 F(U)\overline{G(U)}dU$ and $\|F\| = (F, F)^{1/2}$, respectively.

The set of signals, Γ, is defined as the set of all $f(X)$ which has the Fourier spectrum $F(U)$ satisfying $\|V^{-1}F(U)\|^2 = \frac{1}{(2\pi)^n}\int_{R^n} | F(U) |^2 / | W(U) |^2 dU \le A$, where $A = \frac{1}{(2\pi)^n}A_0$ is a positive constant.

Now, we consider the domain $B_{K,P}$ satisfying

(f) $X_{K,P} \in B_{K,P}$ $(K \in \Lambda, P \in Z^n)$

(g) $B_{K,Q}$ is identical with the parallel translation of $B_{K,P}$ along with the vector $X_{K,Q} - X_{K,P}$.

Moreover, we consider the functions $\psi_{K,P}(X)$ $(K \in \Lambda,$ $P \in Z^n)$ satisfying $\psi_{K,P}(X) = 0$ $(X \notin B_{K,P};\ K \in \Lambda,$ $P \in Z^n)$. We assume that all the $\psi_{K,P}(X)$ are bounded.

For each K, let $f_K(X)$ be the output of the analysis filters $H_K(U)$. Then, the approximation of $f(X)$ is defined by $y(X) = \sum_{K\in\Lambda}\sum_{P\in Z^n} f_K(X_{K,P})\psi_{K,P}(X)$, where $f_K(X_{K,P})$ are the sample values of $f_K(X)$ at $X_{K,P}$ ($K \in \Lambda,\ P \in Z^n$). We call $\psi_{K,P}(X)$ a generalized interpolation function.

Now, let $E(X) = \sup_{f(X)\in\Gamma}\{| e(X) |\}$ and let $S(U, X) =$ $e^{jUX^t} - \sum_{K\in\Lambda}\sum_{P\in Z^n}\psi_{K,P}(X)H_K(U)e^{jUX^t_{K,P}}$. Then, $E(X)$ is expressed as $E(X) = \frac{1}{(2\pi)^n}\sqrt{A_0}[\int_{R^n} | W(U) |^2$ $| S(U, X) |^2 dU]^{1/2}$. Since this equation is almost the same as Eq.(4), we can proceed to similar formulation as before. Hence, we will only present the high light.

(h)Let $\psi_{K,P}(X)$ be the unique optimum interpolation functions minimizing the $E(X)$ and let $\psi_K(X)$ be the functions defined by $\psi_K(X) = \psi_{K,\theta}(X + A_K G^t)$ ($\theta =$ *zero vector in* Z^n). Then, we have $\psi_{K,P}(X) = \psi_K(X -$ $X_{K,P}) = \psi_K[X - (P * T + A_K)G^t]$. Note that these optimum interpolation functions can be realized by shift invariant interpolation filters.

(i)For the conditions (f) and (g), here, $\psi_{K,P}(X)$ do not satisfy the discrete orthogonality, in general. However, when the supports of $H_K(U)$ are included in a Γ_i, the discrete orthogonality as Eq.(11) holds. This situation occurs if $H_K(U)$ are partial differential circuits having the transfer function $\sum_P a_P(ju_1)^{p_1}(ju_2)^{p_2}\ldots(ju_n)^{p_n}$ $(P =$ $(p_1, p_2, \ldots, p_n))$. Since the optimum $y(X)$ satisfies the discrete orthogonality, if the sample values $f_K(X_{K,P})$ are identical with the required discrete initial values, this optimum $y(X)$ satisfies the corresponding discrete initial values and gives a favorable approximation of the solution of the partial differential equation.

(j)If the interpolation functions satisfy Eq.(11), the system can distinguish each of the coefficients in the linear combination of H_K. Since H_K are given firstly, this will be important to radar application.

5. CONCLUDING REMARKS

Although detail is omitted, a linear phase filter bank with 32 paths, the degree 512 and the attenuation 100 dB is obtained by this approximation. Other multi-dimensional numerical programming is future problem.

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