State Space Behavior in Time-varying Biorthogonal Filter Banks

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Abstract— **Using state space representations of biorthogonal filter banks, it is possible to come up with a compact theory for the transition between two time-invariant filter banks. The transition interval depends on the sizes of the common subspaces spanned by the controllability operators of the decomposition filters and by the observability operators of the reconstruction filters. When the respective operators span the same space, the transition can be made arbitrarily short. If it is zero, then the special case of instantaneous transition is reached.**

I. Introduction

The subject of time-varying biorthogonal filter banks has received an increasing attention recently, [1], [2]. Since most signals are not stationary or are of finite length, this is expected. In nearly all applications, the goal is to achieve good uncorrelation between the signals in the different channels on the decomposition side while maintaining smooth transitions in the impulse responses on the reconstruction side. Consider the basic maximally decimated Q-channel FIR filter bank shown in Fig. 1.

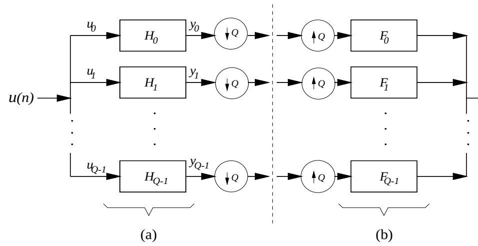

Fig. 1. *A maximally decimated Q-channel multirate filter bank. a) decomposition bank and b) reconstruction bank.*

The action of the filter bank on an infinite signal sequence $u = \{u(n)\}$, $n \in (-\infty, +\infty)$, can be represented using the decomposition reconstruction pair (E, R). For a stationary system E and R are block Toeplitz, with block sizes $Q \times l$ and $l \times Q$, respectively, where Q is the number of channels in the filter bank and l is the length of the filter. In the context of filter banks, the rows of E are called *filter weight vectors* and the columns of R are called the *impulse response vectors.*

State Space Representation

Instead of using the input-output map pair (E, R), the filter bank can also be represented by state space realizations. For the decomposition part E, the state space realization at time instant n maps the present input $u(n) \in l_2^p(\mathbb{Z})$ and the present state $x(n) \in l_2^N(\mathbb{Z})$ to the present output $y(n) \in l_2^Q(\mathbb{Z})$ and the next state $x(n+1)$. Or, in other words, it is the map

$$m(n) : \begin{bmatrix} x(n) \\ u(n) \end{bmatrix} \rightarrow \begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix}, \qquad (1)$$

where the $(N+Q) \times (N+P)$[1] matrix $m(n)$ is explicitly written as

$$m(n) = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}. \qquad (2)$$

Let x represent the state sequence $\{x(n)\}, n \in (-\infty, \infty)$. Likewise, let $y = \{y(n)\}$ and $u = \{u(n)\}$. Then the state space map M is defined as

$$M : \begin{bmatrix} Zx \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix}, \qquad (3)$$

where Z is a shift operator. Similarly, the state space map on the reconstruction side is defined as

$$\bar{M} : \begin{bmatrix} \bar{Z}\bar{x} \\ \bar{y} \end{bmatrix} = \bar{M} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}. \qquad (4)$$

For perfect reconstruction system, $\bar{M}M = I$. This means, for such a system, the state space realization matrices on the respective sides of the filter bank must be such that $\bar{m}(n)m(n) = I$ for all $n \in (-\infty, +\infty)$. If the system is time varying, $m(n)$ is variable with variable dimension $(N(n) + Q(n)) \times (N(n+1) + p(n))$[2]. Then we talk of a time varying filter bank with time varying state space maps $M(m(n))$ and $\bar{M}(\bar{m}(n))$.

II. Transition Between Filter Banks

Let $m_1(a_1, b_1, c_1, d_1)$ and $m_2(a_2, b_2, c_2, d_2)$ represent the state space realizations of the decomposition parts of the shift invariant filter banks $(E_1(e_1), R_1(r_1))$ and $(E_2(e_2), R_2(r_2))$, respectively. The aim is to design an intermediate realization $m(a, b, c, d)$ such that the transitional output functions both on the decomposition and reconstruction sides are minimal[3]. Let the time axis be

[1] Note that for maximally decimated system, $P = Q$ ($m(n)$ is square)

[2] Here $N(n), Q(n)$ and $p(n)$ are the dimensions of $x(n), y(n)$ and $u(n)$, respectively

[3] There is one output vector for each state space realization. The transition duration is measured by the dimension, Q of the output vector of the transitional realization m.

such that m is a realization at $n=0$. i.e.

$$m(n)=\left\{\begin{array}{lcl} m_1 & : & n\in(-\infty,-1] \\ m & : & n=0 \\ m_2 & : & n\in[1,+\infty) \end{array}\right. \tag{5}$$

By definition, the filter weight vector $h(n)$ at time instant n is given by $h(n)=\left[\begin{array}{cc} c(n)\mathcal{C}(n) & d(n)\end{array}\right]$, where $c(n)$ and $d(n)$ are as defined above and $\mathcal{C}(n)$ is the controllability operator corresponding to $m(n)$ [3]. This means that if the filter weight vectors for $n\geq 1$ have to be equal to the stationary values of the overtaking filter, then the transitional realization $m(a,b,c,d)$ should satisfy

$$\mathcal{C}_2\overset{lef}{=}\left[\begin{array}{cc} a\mathcal{C}_1 & b\end{array}\right], \tag{6}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ represent the controllability operators on the decomposition sides of the initial and the final shift invariant filter banks, respectively and $\overset{lef}{=}$ stands for equality after disregarding possible zero columns on the left-most sides of the matrices. As the initial filter is assumed to have been operating stationarily up to $n=-1$, the filter coefficients for $n<0$ are unaffected by the intermediate system m. This means that the filter behaves the same way as the initial stationary system up to $n=-1$.

For the reconstruction filter, the impulse response $g(n)$ at time step n is given by

$$g(n)=\left[\begin{array}{c} \bar{\mathcal{O}}(n-1)\bar{b}(n) \\ \bar{d}(n)\end{array}\right], \tag{7}$$

where $\bar{\mathcal{O}}(n)$ is the observability operator of $\bar{m}(n)$. Clearly, on the reconstruction side, for the impulse responses to agree with the respective stationary values before and after the transition, the intermediate realization $\bar{m}$ must satisfy

$$\bar{\mathcal{O}}_2\overset{top}{=}\left[\begin{array}{c} \bar{\mathcal{O}}_1\bar{a} \\ \bar{c}\end{array}\right], \tag{8}$$

where $\bar{\mathcal{O}}_1$ and $\bar{\mathcal{O}}_2$ are the observability operators on the reconstruction sides of the initial and the final shift invariant filter banks, respectively, and $\overset{top}{=}$ stands for equality after disregarding possible zero rows at the top-most positions of the matrices

A. Solving for m and $\bar{m}$

Let L_1 and L_2 represent the lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Also,let $\mathcal{C}_2(:,1:K_u)$, with $0\leq K_u\leq\min(L_1,L_2)$, be the first K_u columns of $\mathcal{C}_2$ such that, for a certain matrix a, $a\mathcal{C}_1=\left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]$. $\mathbf{0}$ is a zero matrix of appropriate dimensions. Then, if (6) has to be satisfied, we must have

$$a = \left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]\mathcal{C}_1^{\dagger} \tag{9}$$

$$b = \mathcal{C}_2(:,K_u+1:L_2). \tag{10}$$

Similar conditions for $\bar{a}$ and $\bar{c}$ are

$$\bar{a} = \bar{\mathcal{O}}_1^{\dagger}\left[\begin{array}{c}\mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:)\end{array}\right] \tag{11}$$

$$\bar{c} = \bar{\mathcal{O}}_2(K_u+1:L_2,:). \tag{12}$$

For maximally decimated perfect reconstruction system, $m\bar{m}=I\Rightarrow a\bar{a}+b\bar{c}=I$, where I is identity of appropriate dimensions. Substituting the values of a, $\bar{a}$, b and $\bar{c}$ from equations (9) through (12), we get

$$\left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]\mathcal{C}_1^{\dagger}\bar{\mathcal{O}}_1^{\dagger}\left[\begin{array}{c}\mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:)\end{array}\right]+ \mathcal{C}_2(:,K_u+1:L_2)\bar{\mathcal{O}}_2(K_u+1:L_2,:)=I. \tag{13}$$

If we replace the term $\mathcal{C}_1^{\dagger}\bar{\mathcal{O}}_1^{\dagger}$ in (13) with an identity matrix, the left side terms reduce to $\mathcal{C}_2\bar{\mathcal{O}}_2$. For maximally decimated overtaking filter, however, $\mathcal{C}_2\bar{\mathcal{O}}_2=I$. This implies, (13) is true if and only if

$$\left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]P_r\left[\begin{array}{c}\mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:)\end{array}\right]= \left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]\left[\begin{array}{c}\mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:)\end{array}\right], \tag{14}$$

where $P_r=\mathcal{C}_1^{\dagger}\bar{\mathcal{O}}_1^{\dagger}=\bar{\mathcal{O}}_1\mathcal{C}_1$ is a projector[4] into the row and column spaces of $\mathcal{C}_1$ and $\bar{\mathcal{O}}_1$, respectively. Note that if $K_u=0$ the above equation is trivially satisfied with $a,\bar{a}=[\,]$. Soon, we shall discover what this means.

Once $a,b,\bar{a}$ and $\bar{c}$ are determined, the remaining parameters can easily be solved by generating equations from the $\bar{m}m=I$ (perfect reconstruction) and $m\bar{m}=I$ (maximally decimation) relations.

We call (14), the *subspace fitting* relation. That is, by starting from $K_u=\min(L_1,L_2)$ we search for the maximum subspace common to both filters, and then make the transition along this common subspace. The transition characteristics ranges from instantaneous ($K_u=L_1=L_2$) to blocked ($K_u=0$).

B. Lapped, Blocked and Instantaneous Transitions

Proposition II.1: Let $m(a,b,c,d)$ and $\bar{m}(\bar{a},\bar{b},\bar{c},\bar{d})$ represent the state space realizations of the transition system on the decomposition and reconstruction sides, respectively. Then, (6) and (8) are respectively satisfied by non null matrices a and $\bar{a}$ without violating perfect reconstruction if and only if there exists a $K_u>0$ such that the rows of $\left[\begin{array}{cc}\mathbf{0} & \mathcal{C}_2(:,1:K_u)\end{array}\right]$ are all in the space spanned by the rows of $\mathcal{C}_1$ and the columns of $\left[\begin{array}{c}\mathbf{0} \\ \bar{\mathcal{O}}_2(1:K_u,:)\end{array}\right]$ are all in the space spanned by the columns of $\bar{\mathcal{O}}_1$. □

Proof: Perfect reconstruction is preserved if (13) is satisfied by the intermediate realization. For $K_u=0$, the relation is always satisfied as long as the overtaking filter

[4] This can easily be verified by showing $P_r=P_r^2$

is maximally decimated. Nevertheless, $K_u = 0$ means, both a and $\bar{a}$ are null. Thus, if these have to be non-null matrices then K_u must be non-zero.

Consider the relation given in (6). As discussed earlier, this is non-trivially satisfied if we can find a $K_u > 0$ such that $a\mathcal{C}_1 = [\ \mathbf{0}\ \ \mathcal{C}_2(:, 1 : K_u)\ ]$. Since the matrix $a\mathcal{C}_1$ is formed by the linear combinations of the rows of $\mathcal{C}_1$, it always lies in the space spanned by the rows of $\mathcal{C}_1$. This means that (6) is satisfied by a non-null matrix a, if the rows of $[\ \mathbf{0}\ \ \mathcal{C}_2(:, 1 : K_u)\ ]$, $K_u > 0$, are spanned by the rows of $\mathcal{C}_1$. The reverse is also true. That is to say, if there exists a $K_u > 0$ such that the rows of $[\ \mathbf{0}\ \ \mathcal{C}_2(:, 1 : K_u)\ ]$ are in the space spanned by the rows of $\mathcal{C}_1$, we can always express the former as linear combinations of the rows of the latter. In other words, there exists a non-null matrix a for which $a\mathcal{C}_1 = [\ \mathbf{0}\ \ \mathcal{C}_2(:, 1 : K_u)\ ]$. This completes the "if and only if" part. With the same argument, it can be shown that (8) is satisfied by a non-null matrix $\bar{a}$ if and only if the columns of $\begin{bmatrix} \mathbf{0} \\ \bar{\mathcal{O}}_2(1 : K_u, :) \end{bmatrix}$ are spanned by the columns of $\bar{\mathcal{O}}_1$. ■

The meaning of $a, \bar{a} = []$ is that when making transition from the initial to the final stationary filters, states are not transferred in the process. And, we say that the transition is a blocked one. Blocked transition is equivalent to terminating the initial filter and then starting the overtaking one.

When $a, \bar{a} \neq []$, on the other hand, some of the states are transferred to the overtaking filter and we get a lapped transition. K_u is the measure of the degree of overlap at the transition. It indicates the size of the common subspace spanned by the controllability/observability operators of the initial and the overtaking filters. When K_u is maximum the transition interval is minimum. Note that since, $K_u = 0$ trivially satisfies condition (14), we can always set $a, \bar{a} = []$ whenever lapped transition is possible. The nature of transitions characterized by the maximum value of K_u is summarized in Fig. 2.

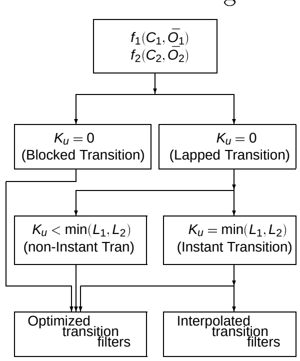

Fig. 2. *Summary of the transition behaviors*

If $K_u = \min(L_1, L_2)$, it can be shown that, under certain conditions, instantaneous transitions both on the decomposition and reconstruction sides are possible. A special class of filter banks under this category is studied in [2]. The filters considered there satisfy $K_u = L_2 = L_1$ and $N_2 = N_1$ and, hence, $m = a$ and $\bar{m} = \bar{a}$. Perfect reconstruction is, therefore, preserved if and only if a and $\bar{a}$ are invertible. It can be shown that [4], for this class of filters, it is possible to make the transition interval arbitrarily small, when it is zero, the special case of instantaneous transition is reached. One interesting feature of this class of filters is that the behavior in the transition period can be controlled using interpolation techniques [4].

III. Extension to two dimensional filters

One can clearly observe that in the formulation of the theory, we didn't restrict ourselves to one dimensional case. In fact, we can directly apply these results to two dimensional filters by properly defining the state space map M. In separable filters, the definition of M is straight forward. One possible choice would be such that filtering is performed column wise first, and then row wise. Transition behaviors in separable two dimensional filters are illustrated in section IV.

IV. Illustrative examples

In this section, we demonstrate the above results for the cases $K_u = \min(L_1, L_2)$ and $K_u = 0$.

A. $K_u = 0$ *(One dimensional filter)*

In this example, the transition behavior between two two channel filter banks of lengths $L_1 = 10$ and $L_2 = 18$ are studied. the structures of both filters are as shown in Fig. 3. In the initial case $f(z) = c_2z^{-2} + c_1z^{-1} - c_1z^1 - c_2z^2$ and $g(z) = c_4z^{-2} + c_3z^{-1} - c_3z^1 - c_4z^2$. The final filter has $f(z) = c_5z^{-1} - c_5z$ and $g(z) = c_6z^{-1} - c_6z$. The transition

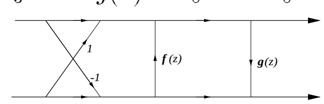

Fig. 3. *A two channel ladder filter bank.*

in the spectral characteristics on the decomposition side and the transition between the impulse responses on the reconstruction side corresponding to the second channel are shown in Fig. 4. From the plots, one can clearly see that the decomposition behavior is preserved by the transition filter weight vectors, and that the impulse responses on the reconstruction side transit in a gentle manner. The other channels behave similarly.

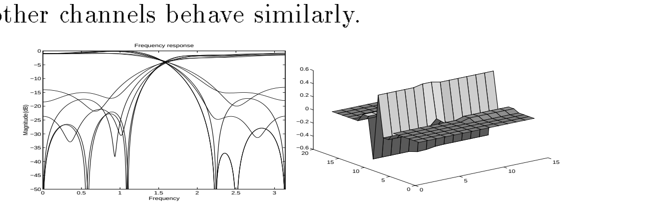

Fig. 4. *Transitions in spectra of filter weight vectors and impulse responses corresponding the second channel.*

B. $K_u = \min(L_1, L_2)$ *(separable two dimensional filter)*

Here, we consider transition from a (2×2)-channel separable two dimensional filter bank to a (4×4)-channel separable two dimensional filter bank. Let $h_2(n)/g_2(n)$ and $h_4(n)/g_4(n)$ represent the decomposition/reconstruction filter pairs of a one dimensional 2-channel and a one dimensional 4-channel filter banks, respectively. Then, a (2×2)-channel and a (4×4)-channel separable filter banks can be constructed from the respective one dimensional counter parts using the tensor products

$$h_{2\times 2}(n,m) = h_2(n)h_2(m) \quad (15)$$
$$h_{4\times 4}(n,m) = h_4(n)h_4(m) \quad (16)$$
$$g_{2\times 2}(n,m) = g_2(n)g_2(m) \quad (17)$$
$$g_{4\times 4}(n,m) = g_4(n)g_4(m). \quad (18)$$

The Fourier transforms of $h_2(n)$ and $h_4(n)$, and the impulse responses $g_2(n)$ and $g_4(n)$ are shown in Fig. 5. Also, the Fourier transforms of $h_{2\times 2}(n,m)$ and $h_{4\times 4}(m,n)$, and the impulse response $g_{2\times 2}(n,m)$ and $g_{4\times 4}(n,m)$ corresponding to the $(1,1)$-channel are given in Fig. 6.

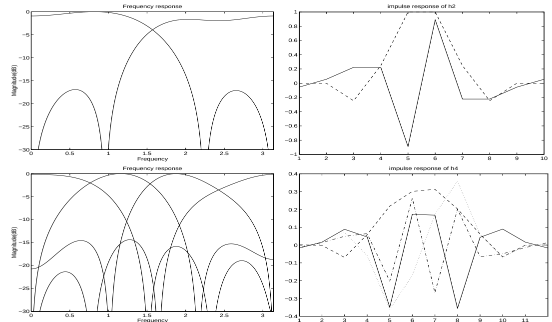

Fig. 5. Fourier transforms of the filter weight vectors and the impulse responses of the 2- and 4-channel filter banks

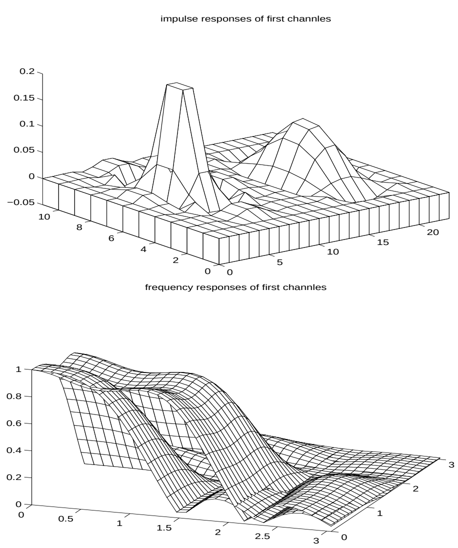

Fig. 6. *Impulse responses and frequency spectrums of the decomposition and reconstruction filters corresponding to the* $(1,1)$*-channels of the initial and final filter banks*

Initially, two 2-channel filter banks are run in time succession as discussed in [4], and then-at time $n = n_o$ say-a constant matrix t is cascaded to the two 2-channel filters and start running as a single 4-channel filter. The take over is instantaneous both on the decomposition and reconstruction sides. That is, the filter banks switch instantly from the top behaviors in Fig. 5 to the bottom ones. Since the two dimensional filters are constructed using relations (15) through (18), they also transit instantaneously from the initial to the final behaviors.

C. Interpolated transition

Instead of appending the constant matrix t at time instant n_o, we now use interpolation techniques - as explained in [4] - along the matrix trajectory $t(n) = g[diag(p_i(n) \mid \lambda_i \mid +(1-p_i(n)))e^{j\gamma_i(n)\theta_i}]g^{-1}$, which takes off at $t(n_o) = I$ and is constant $t(n_f) = t$ from $n = n_f$ on. And similarly for the reconstruction filter: the input matrix starts off from $s(n_o) = I$ and spirals along the trajectory $s(n) = g[diag(q_i(n) \mid \lambda_i^{-1} \mid +(1-q_i(n)))e^{j-\gamma_i(n)\theta_i}]g^{-1}$ to end at constant $s(n_f) = s$ at time instant $n = n_f$. As explained in [4], perfect reconstruction is guaranteed whenever $q_i(n) = \frac{p_i(n)|\lambda_i|}{p_i(n)|\lambda_i|+(1-p_i(n))}$.

The smooth transitions of both the spectral characteristics at the decomposition side and the impulse responses at the reconstruction side are clearly seen in the plots shown in Fig. 7 for channel four. The other channels transit in a likewise gentle way.

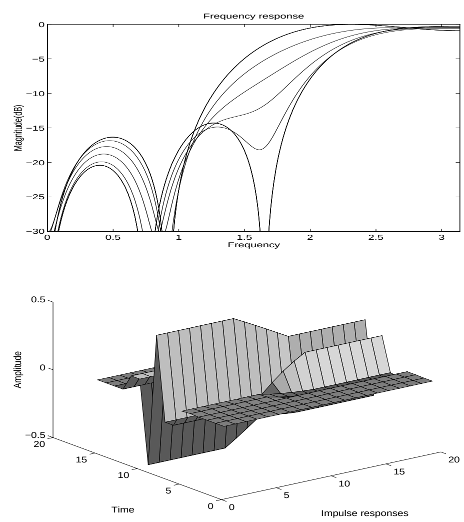

Fig. 7. Smooth transitions in the frequency and impulse responses corresponding to the fourth channel

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