DISCRETE MODELS FOR MULTIDIMENSIONAL SYSTEM SIMULATION

Rudolf Rabenstein
Lehrstuhl für Nachrichtentechnik, Universität Erlangen-Nürnberg
Cauerstrasse 7, D-91058 Erlangen, Germany
Tel: +49 9131 858717; fax: +49 9131 303840
e-mail: rabe@nt.e-technik.uni-erlangen.de

ABSTRACT

Multidimensional continuous systems arising from physical applications with distributed parameters are conventionally modelled by partial differential equations. This paper presents an alternate description by transfer functions based on suitably chosen functional transformations. Signal processing techniques lead to discrete simulation models which are suitable for computer implementation. Numerical results show considerable savings in computer time over existing numerical methods.

1 INTRODUCTION

Multidimensional (MD) systems describe relations between signals depending on two or more independent variables. In many physical and technical applications these are the time and space coordinates. The corresponding systems are given in terms of partial differential equations (PDEs). Typical applications are wave propagation or heat and mass transfer.

In order to simulate the behaviour of these continuous MD systems on a digital computer, one has to turn the PDE description into suitable discrete models containing shift and delay operations instead of derivatives. Traditionally, these models are created by replacing differentiation operators by difference operators (finite difference methods, FDM) or by solving a variational problem on a set of finite elements (finite element methods, FEM). Recently, the use of MD wave-digital filters [2] and functional transformation methods [3, 4] has been proposed. The latter approach consists of two key points:

- employ functional transformations for the independent variables to describe the continuous system by multidimensional transfer functions,
- derive from these transfer functions a discrete model in the form of a multidimensional difference equation suitable for computer implementation.

This approach will be explained here for two kinds of systems given by a hyperbolic and a parabolic PDE, respectively.

2 CONTINUOUS SYSTEM

The first step is the transfer function representation of a MD system. It involves the proper choice of suitable functional transformations for the time and space variables. We show this procedure at first for a hyperbolic problem with simple geometry and boundary conditions and then for a parabolic problem on an arbitrary domain and with more general boundary conditions.

2.1 Hyperbolic problem

As an example for a hyperbolic problem with one spatial dimension we consider a transmission line described by the telegraph equation with initial and boundary conditions (dots and primes denote time and space derivatives)

$$lc\,\ddot{y}(x,t) + (lg+rc)\dot{y}(x,t) + rg\,y(x,t) = y''(x,t) \quad (1)$$

$$y(x,0) = 0, \qquad \dot{y}(x,0) = 0 \quad (2)$$

$$y(x_0,t) = \phi(t), \quad y(x_1,t) = 0 \quad (3)$$

The constants l, c, r, g represent the electrical properties. x_0 and x_1 are the space coordinates of the line terminations. Application of the Laplace transformation with respect to time removes the additional initial conditions and gives the boundary value problem

$$\gamma^2(s)Y(x,s) = Y''(x,s) \quad (4)$$

$$Y(x_0,s) = \Phi(s) \qquad Y(x_1,s) = 0\,. \quad (5)$$

with the propagation function

$$\gamma(s) = \sqrt{(sl+r)(sc+g)} = \frac{1}{v_0^2}(s-\sigma_1)(s-\sigma_2)\,, \quad (6)$$

$$\text{where} \qquad v_0 = \frac{1}{\sqrt{lc}}\,, \quad \sigma_1 = -\frac{r}{l}\,, \quad \sigma_2 = -\frac{g}{c}\,. \quad (7)$$

In order to remove also the boundary conditions, we apply the finite sine transformation [6] to the space variable

$$\bar{Y}(\beta,s) = \int_{x_0}^{x_1} Y(x,s)K(x,\beta)\,dx = \mathcal{T}\{Y(x,s)\} \quad (8)$$

with the transformation kernel

$$K(x,\beta_\mu) = \sin\beta_\mu(x-x_0), \quad \beta_\mu = \pi\frac{\mu-1}{x_1-x_0}\,,\ \mu=1,2,3,\ldots \tag{9}$$

The differentiation theorem for this transformation takes the form

$$\mathcal{T}\{Y''(x)\} = -\beta_\mu^2\bar{Y}(x,\beta_\mu) + \beta_\mu\Phi(s) \tag{10}$$

as is shown by integration by parts. Application of (8) and (10) to (4) gives the algebraic equation

$$\left(\boldsymbol{\gamma}^2(s)+\beta_\mu^2\right)\bar{Y}(\beta_\mu,s) = \Phi(s)\,. \tag{11}$$

which can be solved for the transformed solution $\mathcal{T}\{\mathcal{L}\{y(x,t)\}\} = \bar{Y}(\beta_\mu,s)$ of the PDE (1)

$$\bar{Y}(\beta_\mu,s) = \bar{G}_b(\beta_\mu,s)\Phi(s) \tag{12}$$

$$\text{with} \quad \bar{G}_b(\beta_\mu,s) = \frac{v_0^2\beta_\mu}{(s-s_1(\mu))\,(s-s_2(\mu))}\,. \tag{13}$$

The poles $s_{1/2}(\mu) = \sigma_0 \pm j\omega_\mu$ of $\bar{G}_b(\beta_\mu,s)$ are the roots of $\boldsymbol{\gamma}^2(s)+\beta_\mu^2 = 0$. $\bar{G}_b(\beta_\mu,s)$ is a twodimensional transfer function with the frequency variables s and β_μ. The multiplication of the transfer function with the input $\Phi(s)$ gives the desired transform of the output $\bar{Y}(\beta_\mu,s)$. Thus the frequency domain description (12) of the MD continuous system (1) parallels the conventional use of Laplace transfer functions for onedimensional (1D) systems, e.g. in electrical network theory. The transfer function description (12) contains the same information about the MD system as the PDE (1). Instead of deriving a discrete simulation algorithm from the PDE, e.g. with FDM or FEM, we can also start from the transfer function in order to obtain a discrete simulation model.

The transmission line problem just considered bears some simplifications in order to serve as an introductory example. They result in a boundary value problem (4,5) for which the finite sine transformation (8) is known to yield an algebraic equation containing just the given boundary value functions (11). In more general cases such a transformation is not known a priori and has to be determined from the PDE description of the MD system. This will be shown briefly in the next section.

2.2 Parabolic Problem

Consider the PDE description of a parabolic system

$$\begin{aligned} \dot{y}(\mathbf{x},t) + L\{y(\mathbf{x},t)\} &= v(\mathbf{x},t), & \mathbf{x}\in V \quad, \\ y(\mathbf{x},0) &= y_i(\mathbf{x}), & \mathbf{x}\in V \quad, \\ f_b\{y(\mathbf{x},t)\} &= \phi(\mathbf{x},t), & \mathbf{x}\in S \quad, \end{aligned} \tag{14}$$

with the vector $\mathbf{x}$ of space variables in the domain V, the self adjoint operator L of spatial derivatives, and the boundary operator f_b defined on the surface S. f_b may describe boundary conditions of the first, second, or third kind. The excitation function $v(\mathbf{x},t)$, the initial value $y_i(\mathbf{x})$, and the boundary values $\phi(\mathbf{x},t)$ are assumed to be known. L can be chosen to describe various physical effects, e.g.

$$L\{y(\mathbf{x},t)\} = -\frac{1}{c(\mathbf{x})}(\mathrm{div}(\lambda(\mathbf{x})\mathrm{grad}\, y(\mathbf{x},t))) \tag{15}$$

for heat and mass transfer problems with the material constants c and λ.

Application of the Laplace transformation for the time variable $\mathcal{L}\{y(\mathbf{x},t)\} = Y(\mathbf{x},s)$ turns the initial-boundary value problem into a boundary value problem with the initial value as an additive term:

$$\begin{aligned} sY(\mathbf{x},s) + L\{Y(\mathbf{x},s)\} &= V(\mathbf{x},s) + y_i(\mathbf{x}), & \mathbf{x}\in V \\ f_b\{Y(\mathbf{x},s)\} &= \Phi(\mathbf{x},s), & \mathbf{x}\in S\,. \end{aligned} \tag{16}$$

In order to treat the boundary condition in the same way, we apply a transformation $\mathcal{T}$ for the space variables $\mathbf{x}$ such that the transform of $L\{Y\}$ is expressed by the transform of Y and the known boundary values Φ. This transformation is given by

$$\begin{aligned} \mathcal{T}\{Y(\mathbf{x},s)\} &= \bar{Y}(\beta_\mu,s) = \int_V c(\mathbf{x})Y(\mathbf{x},s)K(\mathbf{x},\beta_\mu)dV \\ \mathcal{T}^{-1}\{\bar{Y}(\beta_\mu,s)\} &= Y(\mathbf{x},s) = \sum_{\mu=1}^{\infty}\frac{1}{N_\mu}\bar{Y}(\beta_\mu,s)\,K(\mathbf{x},\beta_\mu)\,. \end{aligned} \tag{17}$$

$K(\mathbf{x},\beta_\mu)$ and β_μ are the orthogonal eigenfunctions and eigenvalues of the Sturm-Liouville problem

$$L\{K(\mathbf{x},\beta_\mu)\} = \beta_\mu^2 K(\mathbf{x},\beta_\mu)\,, \quad f_b\{K(\mathbf{x},\beta_\mu)\} = 0. \tag{18}$$

The transformation $\mathcal{T}$ is called *generalized Fourier transformation* or *Sturm-Liouville transformation (SLT)* [1] . Application of $\mathcal{T}$ to (16) gives an algebraic equation, which can be solved for the transform of the solution $y(\mathbf{x},t)$ (see [3])

$$\bar{Y}(\beta_\mu,s) = \bar{G}(\beta_\mu,s)\left[\bar{V}(\beta_\mu,s) + \bar{y}_i(\beta_\mu) + \bar{\Phi}_b(\beta_\mu,s)\right] \tag{19}$$

This is the desired transfer function representation, where $\bar{V}(\beta_\mu,s)$, $\bar{y}_i(\beta_\mu)$, and $\bar{\Phi}_b(\beta_\mu,s)$ are the transforms of the excitation, initial and boundary values, respectively. The transfer function

$$\bar{G}(\beta_\mu,s) = \frac{1}{s+\beta_\mu^2}\,. \tag{20}$$

describes the physical behaviour of (14) in the domain of the frequency variables s and β_μ.

3 DISCRETE SYSTEM

In order to derive discrete simulation models we start from the frequency domain description of a continuous system and proceed in two steps.

- Time discretization turns the system with continuous time and space coordinates into a discrete time, continuous space system. We call it a *hybrid system*.

- Space discretization turns the hybrid system into a discrete time, discrete space system or simply a *discrete system*.

For the time discretization, we can apply well known analog to discrete transformations from onedimensional (1D) signal processing like impulse, step, ramp invariant or bilinear transformation.

The space discretization can be performed in two ways. If the discrete frequency parameters β_μ of the SLT are equidistant, then a multiplication in the spatial frequency domain (β_μ) turns into a convolution in the space domain (x). It can be approximated by a discrete convolution performed at discrete values x_n of the space variable. On the other hand, the inverse SLT (17) can also be performed numerically, even if the frequency parameters β_μ are not equally spaced. This is very flexible and numerical effective as long as the series in (17) is rapidly convergent.

3.1 Time discretization

The hybrid systems resulting from time discretization are expected to take samples of the time dependent functions at sampling interval T as input and to deliver samples of the output function y. These sequences are described in the frequency domain by the $\mathcal{Z}$-transformation.

For the hyperbolic problem, the input sequences are given by $\Phi(z) = \mathcal{Z}\{\phi(kT)\}$ whith the discrete time variable k. The frequency domain description of the hyperbolic system is

$$\bar{Y}(\beta_\mu, z) = \bar{G}_b^h(\beta_\mu, z)\Phi(z) \tag{21}$$

where $\bar{G}_b^h(\beta_\mu, z)$ is to be determined from $\bar{G}_b(\beta_\mu, s)$. With respect to time discretization, we can regard this transfer function as a bilinear form in s with constant coefficients. Application of well-known analog to discrete transformations yields bilinear forms in z

$$\bar{G}_b^h(\beta_\mu, z) = \frac{\bar{b}_2(\beta_\mu)z^2 + \bar{b}_1(\beta_\mu)z + \bar{b}_0(\beta_\mu)}{z^2 + \bar{c}_1(\beta_\mu)z + c_0} \tag{22}$$

For example, impulse, step or ramp invariant transformation yield the denominator coefficients $\bar{c}_1(\beta_\mu) = -2\exp(\sigma_0 T)\cos\omega_\mu T$, $c_0 = \exp(2\sigma_0 T)$. The numerator coefficients $\bar{b}_i(\beta_\mu)$ depend on the specific kind of transformation (see [5] for details). The structure of the hybrid system is shown in fig. 1.

Similar considerations for the parabolic system lead to the structure in fig. 2. The feedback coefficient is $\bar{c}(\beta_\mu) = \exp(-\beta_\mu^2 T)$ for the analog to discrete transformations from above. The numerator coefficients for samples of the excitation and boundary value functions are $b_{e,i}(\beta_\mu)$ and $b_{b,i}(\beta_\mu)$.

The hybrid systems shown above resemble first and second order discrete systems with respect to time. Their variables are vectors of infinite length containing

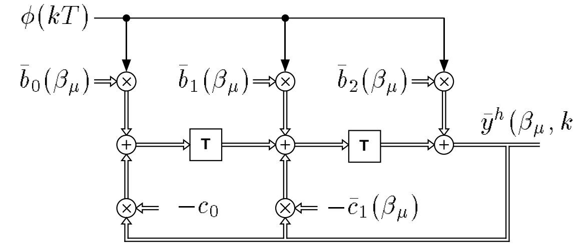

Figure 1: Hybrid system for the hyperbolic problem

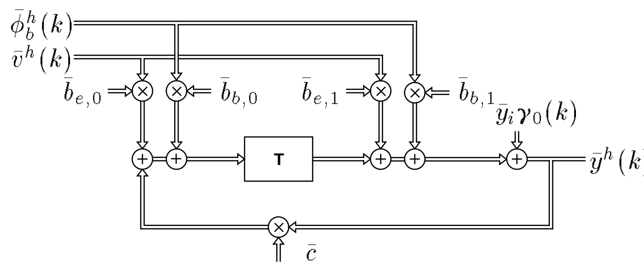

Figure 2: Hybrid system for the parabolic problem (dependence on β_μ omitted)

the discrete SLT coefficients (indicated by double lines). Stability tests from 1D discrete systems can be applied separately for each SLT coefficient and show that the systems in figs. 1 and 2 are inherently stable.

3.2 Space discretization

The simplest way for the discretization of the space variable $\mathbf{x}$ is to implement the hybrid system for a finite number N of SLT coefficients β_μ, $\mu = 1, \ldots, N$ and to evaluate $y(\mathbf{x}_n, k) = \mathcal{T}^{-1}\{\bar{y}^h(\beta_\mu, k)\}$ according to (17) at the discrete points $\mathbf{x}_n$ of interest. However, depending on the boundary conditions, convergence problems similar to Gibb's phenomenon might occur close to the boundary. They can be avoided by separating the output $y^h(\mathbf{x}, k)$ into a part which fulfills the nonhomogeneous boundary conditions $f_b\{y\}$ in (14) and into a part with homogeneous boundary conditions. There exist systematic procedures to determine the first part in closed form. The second part is obtained from a rapidly convergent series [4]. The result is a fully discrete stable MD system. Its inputs are sampled versions of the excitation, initial and boundary values and its output closely resembles the output of the corresponding continuous systems at the sampling points in time and space.

The other way to perform space discretization is to apply the inverse SLT not only to the output signal but to the whole structure of the hybrid system. This poses no problem for signals that are delayed in time, added or multiplied by a constant with respect to μ. However, in general there is no simple space domain expression for the product of two SLT transforms like $\mathcal{T}^{-1}\left\{\bar{c}(\beta_\mu)\bar{y}^h(\beta_\mu, k)\right\}$. In other words, the SLT in the general form of (17) does not have a simple convolution theorem. On the other hand, the situation is different for special cases like the finite sine transform (8). Here

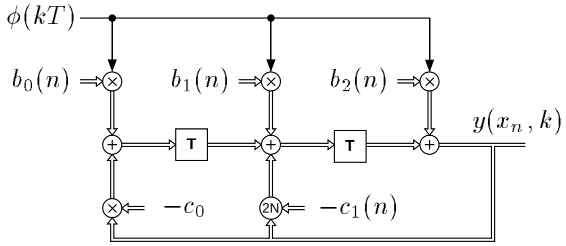

Figure 3: Discrete model for the hyperbolic problem

we can express the inverse transform of a product by the convolution of the corresponding space functions, e.g.

$$\mathcal{T}^{-1}\{\bar{c}_1(\beta_\mu)\bar{y}^h(\beta_\mu, k)\} = c_1(x) * y^h(x, k) \qquad (23)$$

with $c_1(x) = \mathcal{T}^{-1}\{\bar{c}_1(\beta_\mu)\}$. The exact formulation of the convolution requires some modification of the definition range of the spatial functions and is given in [5]. The important point is, that the continuous space convolution in (23) can be approximated by a cyclic discrete convolution ($\textcircled{2N}$) performed on equidistant samples $x_n = nh$. Applying this approach to the hyperbolic problem considered above yields a further advantage. Due to the finite propagation speed of signals on the transmission line, the spatial function $c_1(x)$ will have finite support much less than the length of the line. So the numerical expense for the discrete convolution increases only linear with the number of spatial grid points. Fig. 3 shows the structure of the resulting discrete model for the transmission line. Once the coefficient sequences are known, the simulation consists only of performing the additions, multiplications, delays and the convolution in the feedback path for each time step.

4 NUMERICAL EXAMPLE

Numerical results for the hyperbolic problem are presented in [5]. Here we show the result for a parabolic problem with L according to (15). It describes the dynamic response of the heat flow through a two-layer slab with convection-type boundary conditions (Robins) and without internal sources ($v(\mathbf{x}, t) = 0$). The discrete model consists of the hybrid system of fig. 2 and a numerical inverse SLT with separation of the output.

Fig. 4 compares this discrete system to a standard finite difference method (Crank-Nicolson discretization, LU-factorization) . Shown is the deviation from a reference solution over the number of floating point operations for different numbers of eigenvalues (functional transformation) and different grid sizes (finite difference), respectively. The implementation of the discrete model obviously yields a highly efficient numerical algorithm.

5 SUMMARY

Continuous multidimensional systems, which are conventionally given in terms of PDEs can also be described by transfer functions. Suitable functional transformations for the time and space coordinates are the Laplace and the Sturm-Liouville transformations, respectively. In general, the transformation kernel of the SLT follows from a Sturm-Liouville problem derived from the PDE. The resulting transfer functions are of first order with respect to time for parabolic problems and of second order for hyperbolic problems.

These MD transfer functions are the starting point for the derivation of discrete simulation models. They are well suited for computer implementation since they require only addition, multiplication and delay elements and are free of implicit loops. Furthermore they prove to be far more effective than existing numerical methods.

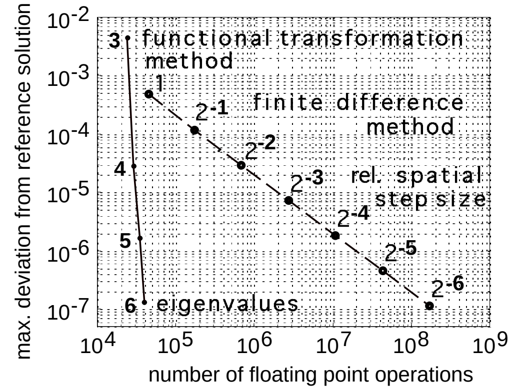

Figure 4: Comparison between functional transformation (——-) and finite difference method (- - - -)

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