CHARACTERISATION OF THE WIGNER-VILLE DISTRIBUTION OF K-NOISE

Miguel A. Rodríguez and Luis Vergara

Dpto Comunicaciones, Universidad Politécnica Valencia
Camino de Vera s/n, 46071 Valencia, Spain
Tel: +34 6 3877300; fax +34 6 3877309
e-mail:mar@dcom.upv.es

ABSTRACT

In this paper we present a statistical characterisation of the Wigner-Ville transform of k-noise. The results show that the positive and the negative values of the Wigner-Ville transform may be separately considered k-distributed random variables with small distribution parameters. The characterisation has been done analytically, but simulations have shown a great agreement with our theoretical model.

1 INTRODUCTION

The detection of signals corrupted by noise is a very common problem in signal processing. A typical example appears when a pulse is emitted and there are expected echoes from the propagation medium, as occurs in radar and ultrasonic non-destructive evaluations. In these situations, the echoes very often appear corrupted by non-Gaussian noise. It is necessary to distinguish between the echoes from the propagation medium and the echoes from anomalies in the medium, this can be done by using the Wigner-Ville transform (WVT) [1]. In these cases the statistical characterisation of the WVT of noise can be very useful.

The k-noise is a type of non-Gaussian noise proposed by Jakeman [2, 3] for modelling the sea clutter in radar applications. More recently, the application of the model to the backscattering noise in ultrasonic non-destructive testing was demonstrated [4, 5, 6, 7, 8, 9]. The k-noise has a k-distributed envelope, having a probability density function (PDF) of the form:

$$p_e(e) = \frac{2b_e}{\Gamma(m_e)}\left(\frac{b_e e}{2}\right)^{m_e} K_{m_e-1}(b_e e) \tag{1}$$

where b_e and m_e are the parameters of the distribution (being m_e greater than 0) and K_{m_e-1} is the (m_e-1)-th order modified Bessel function of third kind. On the other hand the phase will be considered uniformly distributed between 0 and 2π.

Changing the parameter m_e (the parameter b_e is just a scale factor equal to $2\sqrt{m_e}$ when the noise is unit power normalised) in the k-distribution we can consider a wide family of PDFs, ranging from a log-normal distribution for m_e values close to 0, to a Rayleigh distribution for m_e greater than 10. Thus the impulsive character of the k-noise increases as the parameter m_e approaches zero.

The WVT [10, 11, 12, 13] is a bilinear time-frequency analysis tool very often applied to detection problems. In those cases the desired signal is corrupted by noise, thus if we are able to characterise the PDF of the noise WVT, we will be able to design detection tests with controlled probability of false alarm.

The scheme of this paper is the following: in Section 2 we propose an analytical model for the PDF of the k-noise WVT. In Section 3 we obtain a characterisation of the WVT by means of simulations and we compare it with the theoretical predictions of Section 2. Finally we present the conclusions of the paper.

2 ANALYTICAL CHARACTERISATION OF THE WVT OF K-NOISE

The characterisation of the WVT of k-noise is made by using the works of Jakeman [2] and Shankar [7] and by means of three theorems [5]. The three theorems are:

Theorem 1: The random variable (RV) obtained as the product of two independent k-RVs with parameter m_1, is another k-RV with parameter m_2 less than one.

Theorem 2: The addition or difference of two random phases uniformly distributed between 0 and 2π is another random uniformly distributed phase between 0 and 2π.

Theorem 3: The RV generated by the product of two RVs whose distributions are respectively uniform between 0 and 1 and k-type follows another k-distribution.

As will be shown in this next section the positive and negative parts of the WVT of k-noise will separately be k-distributed. For clarity we will follow this notation: the parameters of the k-noise (see (1)) will be m_e and b_e, while the parameters of the positive and negative parts of the k-distributed WVT will respectively be m_+, m_-, b_+ and b_-.

This work has been partially supported by NATO under grant CRG920055

Let $x(n)$ be a k-noise whose WVT we want to characterise. We will consider that $x(n+l)$ and $x(n)$ are independent and identically distributed RVs for any l different to zero.

If we call $a(n)$ to the noise envelope and $\phi(n)$ to the noise instantaneous phase, the expression of the noise will be:

$$x(n) = a(n) \cdot \cos(\phi(n)) \tag{2}$$

Being its analytical signal:

$$z(n) = a(n) \cdot e^{-j\phi(n)} \tag{3}$$

The discrete WVT can be obtained by means of [10]:

$$WV_x(n,f) = \sum_{l=-L}^{L} z(n+l) \cdot z^*(n-l) \cdot e^{-j4\pi fl} \tag{4}$$

Replacing (3) in (4):

$$\begin{aligned} WV_x(n,f) &= \\ &= \sum_{l=-L}^{L} a(n+l) \cdot e^{-j\phi(n+l)} \cdot a(n-l) \cdot e^{+j\phi(n-l)} \cdot e^{-j4\pi fl} = \\ &= \sum_{l=-L}^{L} a(n+l) \cdot a(n-l) \cdot e^{-j\phi(n+l)} \cdot e^{+j\phi(n-l)} \cdot e^{-j4\pi fl} \end{aligned} \tag{5}$$

The statistical characterisation is independent of the value of n, thus we select $n = 0$, without lost of generality

$$\begin{aligned} WV_x(0,f) &= \sum_{l=-L}^{L} a(l) \cdot a(-l) \cdot e^{-j\phi(l)} \cdot e^{+j\phi(-l)} \cdot e^{-j4\pi fl} = \\ &= a^2(0) + \sum_{l=1}^{L} 2 \cdot a(l) \cdot a(-l) \cdot \cos(-\phi(l) + \phi(-l) - 4\pi lf) \end{aligned} \tag{6}$$

If we now define the new variables:

$$\begin{aligned} a'(l) &= 2 \cdot a(l) \cdot a(-l) \\ \phi'(l) &= \phi(-l) - \phi(l) - 4\pi lf \end{aligned} \tag{7}$$

we can write the expression (6) in the form:

$$WV_x(0,f) = a^2(0) + \sum_{l=1}^{L} a'(l) \cdot \cos(\phi'(l)) \tag{8}$$

If $a(l)$ and $a(-l)$ are statistically independent, Theorem 1 demonstrates that $a'(l)$ is k-distributed with some parameter $m_{a'}$.

If there are statistical independence between $\phi(l)$ and $\phi(-l)$, it is obvious from Theorem 2 that $\phi'(l)$ is uniformly distributed between 0 and 2π.

Thus, the equation (8) is the summatory of the real part of complex RVs having k-distributed magnitude and uniformly distributed phase plus the term $a^2(0)$. The characterisation of that summatory (k-envelope and uniform phase) was obtained by Jakeman [2], resulting another k-noise with a new parameter m_c that is the product of the parameter $m_{a'}$ and the number of elements of the summatory, then the equation (11) can be expressed by:

$$WV_x(0,f) = a^2(0) + c \cdot \cos(\phi_c) \tag{9}$$

where ϕ_c is a RV uniformly distributed between 0 and 2π and c is a k-distributed RV with parameter $m_c = L \cdot m_{a'}$ [2]

If ϕ_c is uniformly distributed between π and $-\pi$ the RV $\cos(\phi_c)$ is another RV distributed between -1 and 1.

Now, we divide the WVT into two subtransforms, one for the positive values and the other for the negative ones. This division is convenient because very often the desired signal to be detected is a Gaussian envelope pulse with positive WVT[12]. This positivity implies that in practice we should use a positive threshold for detection purposes, so the probability of false alarm of the detector will be fixed only by the positive values of the WVT of noise and the probability of detection by the negative ones, then

$$WV_x(0,f) = WV(+) - WV(-)$$

where:

$$WV(+) = \begin{cases} c \cdot \cos(\phi_c) + a^2(0) & \text{if } \cos(\phi_c) + a^2(0) > 0 \\ 0 & \text{others} \end{cases}$$

$$WV(-) = \begin{cases} c \cdot (-\cos(\phi_c)) - a^2(0) & \text{if } \cos(\phi_c) + a^2(0) < 0 \\ 0 & \text{others} \end{cases} \tag{10}$$

Theorem 3 demonstrates that the RV resulting from the product of two RVs, one having k-distribution and another a uniform distribution between 0 and 1 is another k-distributed RV. Thus, if $a^2(0)$ could be neglected and $\cos(\phi_c)$ were uniformly distributed, the values different to zero of the subtransforms $WV(+)$ and $WV(-)$ would be k-distributed.

The distribution of $\cos(\phi_c)$ may be approximated by an uniform distribution, this approximation is specially good in the central values of the interval $[-\pi,\pi]$. We have confirmed by means of simulations that the influence of the non-uniformity of $\cos(\phi_c)$ is valueless.

The influence of an additional term $a^2(0)$ in the summatory has been treated by Shankar [7]. The results of Shankar show that the influence in the summatory of k-noise terms of a fix, great and positive additional term implies a bias in the parameters of the model. This point will be treated lately with the presentation of the results.

The conclusion of this section is that the positive and negative values of the WVT (without zeros) of the k-noise can be reasonably supposed k-distributed.

3 EXPERIMENTAL CHARACTERISATION OF THE WVT OF K-NOISE

3.1 Simulations

In order to confirm the previous proposed model, we have made an experimental characterisation.

We have made simulations for four different kinds of noise: k-noise with parameters m_e equal to 0.5, 1 and 3 and Gaussian noise. For every kind of noise we have generated 25600 samples, processed in frames of 100 samples. The noise was uncorrelated and unit power normalised in all cases.

The algorithm utilised for the implementation of the WVT was that proposed by Black and Boashash [10] with the following values: length of the window analysis 15 samples, overlap between windows 14 samples and spectral resolution 16 frequencies.

For each kind of noise we obtain two measures for the positive values and the negative ones of the WVT: the percentage of values, and the parameters m_+ and m_- of the new distributions.

The parameters m_+ and m_- of the new k-distributions have been obtained by using the Raghavan´s method [14]. This technique has been selected because it provides a great accuracy in the estimation of parameters for k-distributions having values of m between 0.2 and 2 [15], that are the expected results from our characterisation.

3.2 Comparison Between the Analytical Model and the Simulation Results

We have proposed above an analytical model for the PDF of the WVT of k-noise. In this point we are going to confirm that the results from the proposed model and from the simulations are coincident.

For the case of our simulations, where the analysis window was 15 samples and the overlap 14 samples. The expression (8) can be written

$$WV_x(0,f) = a^2(0) + \sum_{l=1}^{7} a'(l) \cdot \cos(\phi'(l)) \tag{11}$$

We are going to compare the values obtained from simulations for the selected measures with those ones predicted by the model.

3.2.1 *Percentage of Positive and Negative Values*

From expression (11) we can see that we have one term that is always positive, $a^2(0)$, and seven terms that can be positives or negatives with the same probability. Supposing that the magnitudes of all the terms are similar we can simplify our problem, thus we are going to consider that $WV_x(0,f)$ is positive when the number of positive elements of the summatory is greater than the number of negative ones and vice versa for the negative case. Assuming independence, the number of positive elements in the summatory will follow a binomial distribution with parameter 0.5 and number of elements equal to 7. The mean of this binomial distribution is 3.5 (0.5*7) [16]. Thus we can estimate (remember that $a^2(0)$ is always positive)

$$\text{mean \% of positive values: } \frac{100}{8} \cdot 4.5 = 56.25\%$$

$$\text{mean \% of negative values: } \frac{100}{8} \cdot 3.5 = 43.75\%$$

Table 1 shows the comparison between the theoretical mean percentages and the ones obtained from simulations

	% + theor.	% + simul.	% - theor.	% - simul.
k with $m_e = 0.5$	56.25	58.75	43.75	40.10
k with $m_e = 1$	56.25	58.90	43.75	39.95
k with $m_e = 3$	56.25	58.80	43.75	40.05
Gaussian noise	56.25	58.75	43.75	40.11

Table 1: Percentage of positive and negative values of the WVT of k-noise

We can observe the great agreement that appears between the proposed model and the simulations. The small differences are due to two effects, the first one is the no inclusion of values equal to zero neither in the positive values nor in the negative ones of the theoretical estimate, and the second one is the difference between the magnitudes of $a^2(0)$ and $a'(l) \cdot \cos(\phi'(l))$. In practice the magnitude of $a^2(0)$ is slightly superior than the magnitude of the terms $a'(l) \cdot \cos(\phi'(l))$, then in a real case the percentages of positive values must be slightly superior and the percentage of negative values must be slightly inferior than the theoretical predictions.

On the other hand the percentages of positive and negative values are very independent, in simulations, from the kind of noise, which agrees with the theoretical model.

3.2.2 *Parameters m_+ and m_- of the k-distribution*

From our model the parameter m_+ of the positive part must be greater than the parameter m_- corresponding to the negative one, because of the influence of the term $a^2(0)$. When the resulting value of the WVT of the k-noise is negative the influence of the term $a^2(0)$ must be valueless while when the result is positive this influence must be considered. Then we have two possible kinds of summatories, the first one with seven RVs and the second one with eight RVs. Taking into account that the parameter m is proportional to the number of contributions in the summatory [2], it should be expected a greater parameter value for the positive part of the WVT.

In the next table we show the theoretical parameter m from the proposed model and the parameters m_+ and m_- from simulations. In the theoretical model we suppose that the term $a^2(0)$ of the expression (10) is null.

noise	m	m_+	m_-
k with $m_e = 0.5$	0.52	0.56	0.51
k with $m_e = 1$	0.56	0.64	0.57
k with $m_e = 3$	0.585	0.71	0.60
Gaussian noise	0.59	0.74	0.65

Table 2: Theoretical estimation of parameter m of WVT for different kinds of noise and comparison with results from simulations

Comparing the results of Table 2, theoretical m (first column) with the parameters m_+ and m_- of the positive and negative part of the WVT, we can observe again a confirmation of our proposed model. The proposed model works better for the cases of noise having parameter m_e equal to 0.5 and 1, and it is specially good for the negative values of the WVT. For the negative values the influence of the term $a^2(0)$ must be very low, then the assumption of $a^2(0)$ equal to zero is almost true. In the case of the positive values, the influence of the term $a^2(0)$ must produce a bias in the parameter of the model [7].

4 CONCLUSIONS

The main contribution of this paper is a model for the PDF of the WVT of k-noise. The model has been obtained by means of analytical methods, but it has been validate by simulations. Simulations show a great agreement with the proposed model predictions.

This analytical characterisation allows the design of signal detectors in a k-noise background using the WVT. Based on this characterisation it has been designed a detector for Gaussian envelope pulses [17].

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