ALGEBRAIC LATTICE REALIZATION OF PASSIVE TRANSMISSION LINE SYSTEMS

Yoshimi Monden, Masayasu Nagamatsu, and Satoru Okamoto

Department of Mathematics and Computer Science
Interdisciplinary Faculty of Science and Engineering
Shimane University
Nishikawatsu, Matsue, 690 Japan

ABSTRACT

In this paper, firstly, the Schur-Cohn test known as an algebraic stability test of discrete-time linear systems is presented as a "lossless bounded realness test by lossless bounded real lattice realization" of a given real rational transfer function on the unit disk. Then, by characterizing a discrete model of piecewise constant passive transmission line in terms of a set of physical system parmeters, it is extended to an algebraic algorithm for "bounded realness test by bounded real realization" of a certain class of rational transfer functions, which are general enough to cover almost actual passive transmission lines.

1 Introduction

The Schur-Cohn (SC) test first appeared as an algebraic criterion for location of zeros of polynomials in the unit disk and has played an important role in testing stability of discrete-time linear systems. On the other hand, Levinson-Durbin algorithm, first worked out as a recursive algorithm to solve efficiently the Wiener's linear prediction problem, has found wide application in seismic and speech analyses and underlain a variety of problems such as linear prediction, orhtogonal polynomial, inverse scattering and spectral analysis.

In spite of their apparent similiarity in form, it was not until the last seventies that they came to be recognized as different views of the same structure of lossless tramnsmission lines and it is still scarecely known that the SC test could also be regarded not only as a lossless bounded realness criterion but as a lossless bounded real (LBR) realization algorithm for real rational transfer functions on the unit disk. These observations show that lossless bounded realness test and LBR realization go side by side and share a common algebraic algorithm, so called SC test for their solution.

A number of works have appeared in this context, some exclusively associated with algebraic criteria and some with realization problems, and have not yet succeeded in bounded real (BR) extension of the SC test to general passive transmission lines.

- Siljak was the first who presented complete algebraic criteria for positive realness of rational functions both on the unit disk and on the half plane. But he did not show its relation to realization problems [1].
- Vaidyanathan proposed a recursive method for BR design of digital filters with the aid of a digital two-pair extraction method [2], [3]. However, strictly speaking, it is not a finite algebraic algorithm but an infinite analytic one in the sense one has to compute zeros and/or maxima of polynomials at its recursion steps.
- Bruckstein and Kailath considered realization problem in the context of inverse scattering problem and showed close relationship between the inversion algorithm and the Schur's algorithm only for lossless transmission lines[4].
- Nagamatsu et al. presented a discrete model for lossy nonuniform acoustic tubes in terms of local areas and loss factors and proposed a recursive method for estimating these parameters based on multivariate linear prediction [6]. But it lacked in generality and completeness.

Main purpose of this work is to integrate these criteria and algorithms into a single algebraic algorithm for "bounded realness test by BR realization" of given real rational functions on the unit disk. To attain that end, the SC test is first presented as a "lossless bounded realness test by LBR lattice realization" of a given real rational transfer function on the unit disk. Then, by characterizing a discrete model of piecewise constant passive transmission line in terms of a set of physical system parmeters, it is extended to an algebraic algorithm for "bounded realness test by BR realization" of a certain class of rational transfer functions, which are general enough to cover those of almost actual transmission lines.

2 An Algebraic Lattice Extraction Mehtod

In this paper, We consider a linear system characterized by a transfer function of N-th order $H_N(z) =$

$B_N(z)/A_N(z)$ for relatively prime real polynomials $A_N(z)$ and $B_N(z)$ of the same degree N

$$A_N(z) = A_{N0} + A_{N1}z + ... + A_{NN}z^N$$
$$B_N(z) = B_{N0} + B_{N1}z + ... + B_{NN}z^N$$

which can be regarded as the input and the output of our system. In the sequel, for the sake of notational convenience, we will use "z" instead of "z^{-1}" to denote a unit-time delay. Then the transfer function $H_N(z)$ can be realized by an algebraic lattice extraction method as indicated in Figure 1, successively extracting an elementary lattice section from $H_n(z)$ and ending up with a constraining transfer function of (n-1)-th order $H_{n-1}(z) = B_{n-1}(z)/A_{n-1}(z)$ [2],[3]. The elementary lattice section is described either by a scattering matrix $P_n(z)$

$$\begin{bmatrix} A_n(z) \\ B_{n-1}(z) \end{bmatrix} = P_n(z) \begin{bmatrix} A_{n-1}(z) \\ B_n(z) \end{bmatrix} \quad (1)$$

or equivalently by a transfer matrix $Q_n(z)$

$$\begin{bmatrix} A_{n-1}(z) \\ B_{n-1}(z) \end{bmatrix} = Q_n(z) \begin{bmatrix} A_n(z) \\ B_n(z) \end{bmatrix} \quad (2)$$

where the elements of these system matrices are linked through the following relations

$$\begin{aligned} (Q_n)_{11} &= (P_n)_{11} - (P_n)_{12}(P_n)_{21}/(P_n)_{22} \\ (Q_n)_{12} &= (P_n)_{12}/(P_n)_{22} \\ (Q_n)_{21} &= -(P_n)_{21}/(P_n)_{22} \\ (Q_n)_{22} &= 1/(P_n)_{22} \end{aligned}$$

or its inverse ones.

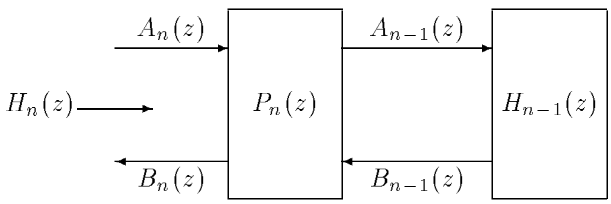

Figure 1: A lattice extraction method

We will make extensive use of transfer matrices and make special choice of their parameters as

$$Q_n(z) = \frac{1}{f_n} \begin{bmatrix} B_{nn} & -A_{nn} \\ -g_n B_{n0} z^{-1} & g_n A_{n0} z^{-1} \end{bmatrix}$$

so that the polynomials $A_n(z)$ and $B_n(z)$ are reduced in degree by eliminating their first and last coefficients, as is the case with the SC test. Then the transfer functions $H_n(z)$ and $H_{n-1}(z)$ are linked through linear fractional transformations of complex analysis

$$\begin{aligned} zH_{n-1}(z) &= g_n[A_{n0}H_n(z) - B_{n0}] \\ &\quad /[-A_{nn}H_n(z) + B_{nn}] \quad (3) \\ H_n(z) &= [B_{nn}zH_{n-1}(z) + g_n B_{n0}] \\ &\quad /[A_{nn}zH_{n-1}(z) + g_n A_{n0}]. \end{aligned}$$

For the sake of completeness, let us cite some definitions concerning lossless bounded realness or bounded realness of transfer functions and matrices as follows:

Definition 1: Let $Q_n(z)$ be a transfer matrix whose elements are real rational functions of z with no poles on the unit disk $|z| \leq 1$. Then $Q_n(z)$ is said to be BR(LBR) if it satisfies $Q_n(z)^* J Q_n(z) \leq J (= J)$, $|z| = 1$ for J=diag[1, -1].

Definition 2: Let $H_n(z)$ be a real rational transfer function of z with no poles on the unit disk $|z| \leq 1$. Then $H_n(z)$ is said to be BR(LBR) if it satisfies $|H_n(z)| \leq 1 (= 1)$ on the unit circle $|z| = 1$.

We are now in a position to describe BR(LBR) realization of a rational transfer function on the unit disk. Introducing common and rating factors f_n, g_n of $Q_n(z)$ respectively and noting that the constraining transfer function $H_{n-1}(z)$ is not affected by these factors upto constant factor, we are led to an important concept of BR(LBR) realizability of transfer matrices and functions.

Definition 3: A transfer matrix $Q_n(z)$ of an n-th elementary lattice is called BR(LBR) realizable, if there exists some non-zero constants f_n and g_n such that $Q_n(z)$ can be made BR(LBR).

With the aids of these definition, we are able to recursively define BR(LBR) realizability of transfer functions, which will play a crucial roll in this paper.

Definition 4: A transfer function $H_N(z)$ in Figure 1 is called BR(LBR) lattice realizable, if $Q_N(z)$ is BR(LBR) realizable and $H_{N-1}(z)$ is also BR(LBR) lattice realizable.

Therefore, BR(LBR) realizability of $H_n(z)$ can be reduced to that of a elementary lattice $Q_n(z)$ and a residual transfer function $H_{n-1}(z)$ only by using the first and the last coefficients of $A_n(z)$ and $B_n(z)$. Therefore once this BR(LBR) realizability condition is met for the transfer function $H_n(z)$ of order n, we use the transfer matrix $Q_n(z)$ to extract the associated BR(LBR) lattice from $A_n(z)$ and $B_n(z)$ and put ourselves in the same position as before with a BR(LBR) transfer function $H_{n-1}(z)$ of order $n-1$ given by a polynomial fraction of $A_{n-1}(z)$ and $B_{n-1}(z)$ of degree $n-1$. We can continue in this way, successively testing the BR(LBR) realizability condition and extracting its associated elementary lattice to obtain a cascaded BR(LBR) realizaton of a given rational transfer function.

3 LBR Lattice Realization and the SC Test

In this section, we will show the equivalence of LBR lattice realizability and lossless bounded realness of a real rational transfer function and present the SC test as a LBR lattice realizability criterion.

The following lemmas and theorem are concerned with LBR realizability of $Q_n(z)$ and $H_n(z)$ respectively, and are easy exercises of *Definition 3, 4*.

Lemma 1: A transfer matrix $Q_n(z)$ is LBR realizable iff the following relations hold:

$$A_{n0}^2 > A_{nn}^2, \; B_{n0} = \pm A_{nn}, \; B_{nn} = \pm A_{n0}.$$

Proof: The desired result readily follows from a set of relations

$$\begin{aligned} A_{n0}^2 &> A_{nn}^2 \\ A_{n0}^2 - A_{nn}^2 &= B_{nn}^2 - B_{n0}^2 \\ A_{n0}B_{n0} &= A_{nn}B_{nn} \end{aligned}$$

which result from a conditon $Q_n(z)$ is LBR

$$\begin{aligned} g_n^2 A_{n0}^2 - A_{nn}^2 &= f_n^2 \\ B_{nn}^2 - g_n^2 B_{n0}^2 &= f_n^2 \\ g_n^2 A_{n0}B_{n0} - A_{nn}B_{nn} &= 0 \end{aligned}$$

for some non-zero f_n and $g_n = \pm 1$.

Lemma 2: A transfer function $H_N(z)$ of N-th order in *Figure 1* is LBR realizable iff it holds for $n = N, ..., 1$

$$A_{n0}^2 > A_{nn}^2, \; B_{n0} = \pm A_{nn}, \; B_{nn} = \pm A_{n0}.$$

Proof: Obvious from recursive application of *Lemma 1* to *Definition 4* together with a relation $H_0(z) = B_{00}/A_{00} = 1$.

Finally, the following theorem establishes the equivalence between LBR realizability and lossless bounded realness of a real rational transfer function and allows us to regard the SC test as their equivalent.

Theorem 1: A real rational transfer function $H_N(z)$ is LBR iff it is LBR realizable.
Proof: It suffices to show that $H_N(z)$ is LBR if and only if both $Q_N(z)$ and $H_{N-1}(z)$ are LBR. Since the "if" part is obvious, we have only to show the "only if" part. Owing to the Jensen-Nevanlinna formula, a real rational transfer function $H_N(z) = B_N(z)/A_N(z)$ is LBR iff it holds 1) $A_N(z)$ has all its zeros outside the unit disk and 2) $B_{Nk} = \pm A_{N,N-k}$ for $k = 0, 1, ..., N$, which in turn guarantee the existence of a LBR $Q_N(z)$. The lossless bounded realness of $H_{N-1}(z)$ results from a couple of recurrence relations

$$\begin{aligned} A_{N-1}(z) &= f_N[B_{NN}A_N(z) - A_{NN}B_N(z)] \\ zB_{N-1}(z) &= f_N[A_{N0}B_N(z) - B_{N0}A_N(z)] \end{aligned}$$

by taking into account the following identities

$$B_{N-1,k} = \pm A_{N-k,N-k-1}, \quad k = 0, 1, ..., N-1$$

and noting that $A_{N-1}(z)$ has all its zeros outside the unit disk due to Rouche's theorem.

It is in this general setting that the SC test could be considered as a LBR realizability condition or equivalently a lossless bounded realness criterion.

4 BR Lattic Realization of Transmission Lines

BR realization is not so tractable as LBR one, because bounded realness of $H_N(z)$ does not necessarily mean that of $Q_N(z)$ and $H_{N-1}(z)$. However the converse is still alive, because

$$\begin{aligned} |A_{N-1}(z)|^2 - |B_{N-1}(z)|^2 &\leq |A_N(z)|^2 - |B_N(z)|^2, \\ |B_{N-1}(z)|^2 &\leq |A_{N-1}(z)|^2 \end{aligned}$$

for $|z| = 1$ imply $|B_N(z)|^2 \leq |A_N(z)|^2$ for $|z| = 1$, which means $H_N(z)$ is bounded real. With the aids of this converse, we will extend the aforementioned LBR realizability condition to a BR one.

4.1 A Discrete model of passive transmission lines

Before going on BR lattice realization of rational transfer functions of transmission lines, we will propose a discrete model of piecewise constant passive transmission lines. This model is composed of cascaded series of uniform lattice sections, characterized by their "lossless boundary scattering"without delay and "absorptive inside propagation" with the same isotropic delay, and is general enough to cover almost actual passive transmission-lines.

To be more specific, let us introduce a quadruple of physical system parameters, right and left reflection coefficients $\kappa_n^{\pm}$ and right and left transmission coefficients $\tau_n^{\pm}$ to model "boundary scattering", which are further reduced to a single parameter κ_n owing to lossless scattering without delay such that

$$\begin{aligned} \kappa_n^+ &= -\kappa_n^- = \kappa_n \\ \tau_n^+ &= \tau_n^- = \sqrt{1-\kappa_n^2} \end{aligned}$$

and also a couple of system parameters, right and left absorption coefficients $\eta_n^{\pm}$ to model "absorptive inside propagation".

Then noting each section has the same isotropic delay and applying a delay-transfer rule to "$z^{\frac{1}{2}}$", we are led to an n-th transfer matrix

$$Q_n(z) = \frac{1}{\sqrt{1-\kappa_n^2}} \begin{bmatrix} \eta_n^+ & -\kappa_n\eta_n^+ \\ -(\kappa_n/\eta_n^-)z^{-1} & (1/\eta_n^-)z^{-1} \end{bmatrix}$$

for $n = N, ..., 1$. Furthermore, defining $A_n(z)$ and $B_n(z)$ in terms of $A_0(z)$ and $B_0(z)$ as

$$\begin{bmatrix} A_n(z) \\ B_n(z) \end{bmatrix} = Q_n^{-1}(z) \cdots Q_1^{-1}(z) \begin{bmatrix} A_0(z) \\ B_0(z) \end{bmatrix} \quad (4)$$

we can easily obtain the equation (2) for $n = N, ..., 1$. This shows that a transfer matrix $Q_n(z)$ of an n-th elementary lattice of our model is a bounded real matrix satisfying the following relation:

$$Q_n(z)_{11}Q_n(z)_{21} = Q_n(z)_{12}Q_n(z)_{22}.$$

4.2 An algebraic algorithm for BR realization

Let us begin with a lemma on bounded realness of a transfer matrix $Q_n(z)$ and only cite it without proof because of shortage of space.

Lemma 3: A transfer matrix $Q_n(z)$ is bounded real iff the following relations hold:

$$\begin{aligned} &g_n^2 A_{n0}^2 > A_{nn}^2 \\ &g_n^2 (A_{n0}^2 - B_{n0}^2) \geq A_{nn}^2 - B_{nn}^2 \\ &g_n^2 (A_{n0} \pm B_{n0})^2 \geq (A_{nn} \pm B_{nn})^2 \end{aligned}$$

Lemma 4: A transfer matrix $Q_n(z)$ is BR realizable, iff $A_{n0}^2 > B_{n0}^2$ for $n = N, ..., 1$.
Proof: Obvious from *Lemma 3* because there does not exist g_n if the above inequality does not hold.

The following theorem is concerned with BR realizability of a given BR transfer function $H_N(z)$ and is obvious from *Definition 4* and *Lemma 4*.

Theorem 2: A real rational function $H_N(z)$ is strictly BR realizable, iff it holds

$$A_{n0}^2 > A_{nn}^2, \qquad A_{n0}^2 > B_{n0}^2$$

under the minimal choice of g_n for $n = N, ..., 1$ and $B_{00}^2 < A_{00}^2$.

For rational transfer functions in **4.1**, this BR realizability condition also comprises their "bounded realness test by BR realization" in the following manner:

Theorem 3: A real rational function $H_N(z)$ belonging to the class in **4.1** is not only strictly BR realizable but also strictly BR, iff it holds

$$\begin{aligned} A_{n0}^2 > A_{nn}^2, &\qquad A_{n0}^2 > B_{n0}^2 \\ A_{n0}A_{nn} &= B_{n0}B_{nn} \end{aligned}$$

under the special choice of $g_n = B_{nn}/A_{n0}$ for $n = N, ..., 1$ and $B_{00}^2 < A_{00}^2$.
Proof: It suffices to show inequalities $A_{N0}^2 > A_{NN}^2$ and $A_{N0}^2 > B_{N0}^2$ as well as bounded realness of $H_{N-1}(z)$ if $H_N(z)$ is bounded real. Both inequalities readily follow from the holomorphy of $H_N(z)$ and the maximum modulus principle. Bounded realness of $H_{N-1}(z)$ can be easily deduced by reducing the linear fractional transformation (3) to

$$\begin{aligned} zH_{N-1}(z) = & [H_N(z) - (B_{N0}/A_{N0})] \\ & /[1 - (B_{N0}/A_{N0})H_N(z)] \end{aligned}$$

by the use of

$$g_N = B_{NN}/A_{N0}, \qquad B_{N0}/A_{N0} = A_{NN}/B_{NN}$$

and following the similar augument in the proof of *Theorem 1*.

5 Conclusions

In this paper, we presented a LBR realization problem in a more general setting, and showed the SC test known as a stability criterion of discrete time systems could also be regarded as "lossless bounded realness test by LBR realization" of real rational transfer functions on the unit disk. And then we have extended the LBR realizability condition to a bounded real one.

Our realization method has many advantages compared with the other ones. Firstly, our method is algebraic based upon a euclidian algorithm using only arithmetic operations of polynomial coefficients, while Vaidyanathan's one is analytic in nature because it requires to find out maximum absolute points of transfer functions at each step of the two-pair extraction procedure [2],[3]. Secondly, by preserving the lattice structure of general transmission lines and providing direct relations to their physical properties, ours is amebable to their physical realization in terms of the physical system parameters common to scattering systems [6].

Although we have not yet succeeded in establishing complete relationship between BR realizability and bounded realness of real rational functions on the unit disk, we conclude this paper by saying that these algebraic criteria will play a very important roll in not only realization problems but also axiomatic system theory only by using elemantary arithmetic operations [5],[6].

References

[1] D.D. Siljak, New Algebraic Criteria for Positive Realness, *J. Franklin Inst.*, vol.291, p.109, 1971, Routh Algorithm: A Centennial Survey, *SIAM REVIEW*, vol.19, p.472, 1977.

[2] P.P. Vaidyanathan and S.K. Mitra, Low Passband Sensitivity Digital Filters: A General Viewpoint and Systhesis Procedure, *Proc. IEEE*, pp. 404-423, 1985.

[3] P.P. Vaidyanathan, A Unified Approach to Orthogonal Digital Filters and Wave Digital Filters, Based on LBR Two-Pair Extraction, *IEEE Trans. Circuits Syst.*, vol.32, pp. 673-686, 1985.

[4] A.M. Bruckstein and T. Kailath, Inverse Scattering for Discrete Transmission-Line Models, *SIAM REVIEW*, vol.29, No.3, p.359, 1987.

[5] Y. Monden, M. Nagamatsu and H. Kayano, How Does Inverse Scattering Problem Encounter with Digital Signal Processing, *IEICE Journal*, vol.75, no.1, pp. 66-75, 1992.

[6] M. Nagamatsu, S. Okamoto and Y. Monden, Symultaneous Estimation of Both Area- and Loss-Functions of Lossy Nonuniform Acoustic Tubes, *Signal Processing V*, pp. 485-488, 1990, North-Holland.