Invariance properties of integral transforms of images.

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ABSTRACT

In this paper previous results on invariance coding are extended in two ways: 1) by proving that there exists a formal relation between the kernel of an integral transform "invariant in the strong sense" and the eigenfunctions of the operator of the transformation 2) by showing that necessary and sufficient conditions for invariance with respect to *one-parameter Lie transformation groups* can hold for a class of *two-parameters transformation groups*, and by providing a procedure to compute an integral transform "invariant in the strong sense" with respect to these transformations.

INTRODUCTION

The problem of invariance is central to pattern recognition: in a fundamental sense invariant recognition is precisely what a recognition system must be able to achieve. The ability of any visual system to perform invariant to a given transformation is determined by the way the visual input is encoded, or internally represented, by the system. In the Cartesian domain (x, y) an image is defined by a function from R^2 to R, $f : x, y \rightarrow z$, where $z = f(x, y)$ is the light intensity at the point x, y. Let T_a be a one-parameter Lie transformation group, with parameter a, acting on f; the transformed image is given by $T_a f(x, y) = f(T_a x, T_a y)$. In general $T_a f \neq T_{a'} f$ if $a \neq a'$, and so this representation is not invariant. Invariant coding deals with the problem of finding representations of the pattern, in some space (u, v), that are invariant under certains transformations and preserve the uniqueness of the representation. This type of invariance has been called "invariance in the strong sense" (Ferraro, 1992). An integral transform

$$\mathcal{G}(f) = G(u,v) = \int\int f(x,y)k(x,y;u,v)dxdy = A(u,v)\exp[i\phi(u,v)] \quad (1)$$

is said to be invariant in the strong sense, under the action T_a on f, if $A(u, v)$ is constant and the phase ϕ varies in a simple additive way (Ferraro and Caelli, 1988, Ferraro, 1992), that is

$$\mathcal{G}(T_a f) = G(u,v) = A(u,v)\exp\{i[\phi(u,v) + ua]\}, \quad (2)$$

so that the constancy of $A(u, v)$ ensures invariance and ϕ encodes the transformational state. In general, it has been proven (Ferraro and Caelli, 1988, Ferraro, 1992) that if an image f is trasformed by the action of two one-parameter Lie groups of transformation T_a, S_b, an integral transform of f,invariant in the strong sense, exists if and only if X_a, X_b, vector fields of T_a and S_b respectively, commute and are linearly indepedent (the commutativity of X_a, X_b implies the commutativity of T_a, S_b). Furthermore it has been shown that the invariant integral transform is of the form

$\mathcal{G}(f) =$

$$\int\int f(x,y)\exp\{-i[u\eta+v\xi]\}|J(\eta,\xi;x,y)|dxdy$$

$$= \int\int f(x,y)\exp\{-i[u\eta+v\xi]\}d\eta d\xi, \qquad (3)$$

where η, ξ are canonical coordinates of T_a and S_b, (that is, in the coordinate system (η,ξ) the actions of T_a and S_a are translations) and $|J(\eta,\xi;x,y)|$ is the Jacobian determinant of the change of coordinates $(\eta,\xi) \rightarrow (x,y)$. Note that in the first integral of equation 3 η,ξ are functions of x,y, whereas in the second they are independent variables. Let $k(x,y,;u,v)$ be the kernel of the transformation when the integration is performed in the x,y domain and let $k(\eta,\xi;u,v)$ denote the kernel when it is carried out in the η,ξ domain. It is immediate that $k(x,y,;u,v) = |J(\eta,\xi;x,y)|k(\eta,\xi;u,v)$ For instance, let T_a be a rotation and let S_b be a dilation. The canonical coordinates are $\eta(x,y) = \tan^{-1}(y/x)$, $\xi(x,y) = 1/2\lg(x^2+y^2)$ and the Jacobian determinant is $\left(x^2+y^2\right)^{-1}$ (Ferraro and Caelli, 1988, Ferraro, 1992).

TRANSFORMATIONAL PROPERTIES OF KERNELS

From equation 3 it is clear that invariance does not depend on the image *per se* but it is a property of the representation of the image, that, in our case, is determined by the kernel $k(x,y;u,v)$ of the integral transform. Then it is of interest to determine which conditions a kernel must satisfy for the corresponding integral transform to be invariant in the strong sense. A necessary and sufficient condition is established by the following Proposition:

Proposition 1. *An integral transform $\mathcal{G}$ is invariant in the strong sense with respect to a one-parameter Lie transformation group T_a, with parameter a, if and only if its kernel $k(x,y;u,v)$ is such that*

$T_a k(x,y;u,v) =$

$$|J(x,y;x',y')|\exp(-iua)k(x,y;u,v), \qquad (4)$$

where $|J(x,y;x',y')|$ is determinant of the Jacobian of the transformation $(x,y) \rightarrow (x',y')$, with $x' = T_a x,\ y' = T_a y$.

Proof. Preliminarily note that $k(x,y;u,v) = k(T_{-a}x', T_{-a}y';u,v) = T_{-a}k(;x',y';u,v)$.

$\Rightarrow$ The integral transform $\mathcal{G}$ is invariant in the strong sense by hypothesis, then

$\mathcal{G}[T_a f(x,y)] =$

$$\int\int f(x',y')k(u,v;x,y)dxdy = \qquad (5)$$

$$\int\int f(x',y')k(T_{-a}x', T_{-a}y';u,v) \mid J \mid dx'dy' =$$

$$\exp(iua)\int\int f(x',y')k(x',y';u,v)dx'dy',$$

where $|J| = \mid J(x,y,;x',y') \mid$. Since the above relation must hold for any f, it follows that

$\exp(iau)k(x',y';u,v) =$

$$\exp(iau)T_a k(x,y;u,v) =$$

$$\mid J(x,y;x',y') \mid k(x,y;u,v). \qquad (6)$$

$\Leftarrow$ From equation 4 it follows that

$\mathcal{G}[T_a f(x,y)] =$

$$\int\int f(x',y')k(x,y;u,v)dxdy = \qquad (7)$$

$$\int\int f(x',y')k(T_{-a}x', T_{-a}y';u,v)|J|dx'dy'.$$

By applying again equation 4, equation 7 can be written as

$\mathcal{G}[T_a f(x,y)] =$

$$\exp(iua)\int\int f(x',y')k(x',y';u,v)dx'dy',$$

and hence $\mathcal{G}[T_a f] = \exp(iua)\mathcal{G}[f]$.

From Equations 5 and 7 it is clear that a transformation of the image f by the action of T_a with parameter value a is equivalent to transformation of the kernel by T_a with parameter value $-a$.

Corollary 1. *An integral transform $\mathcal{G}$ is invariant in the strong sense with respect to an area preserving(respectively linear), one-parameter, Lie transformation group T_a, if and only if the modulus $\mid k(x,y;u,v) \mid$ of its the kernel is invariant with respect to T_a (respectively, invariant but for a constant scaling factor.*

Proof. It is enough to recall that if T_a is area preserving then $|J(x,y;x',y')| = 1$ and that if T_a is linear then $|J(x,y;x',y')| = c$, where c is a constant.

Then, for $\mathcal{G}(f)$ to be invariant with respect to a pair of transformation groups T_a and S_b, the kernel $k(x,y;u,v)$ must be such that for T_a equation 4 holds, and analogously for S_b,

$$S_b k(x,y;u,v) = |J(x,y;x',y')| \exp(-ivb) k(x,y;u,v). \quad (8)$$

Proposition 2. *Let T_a, S_b be two one-parameter Lie transformation groups. An integral transform $\mathcal{G}$ is invariant, in the strong sense, with respect to T_a, S_b if and only the kernel of $\mathcal{G}$ is such that equations 4 and 8 hold.*

Proof. The condition is obviously necessary. To prove that is also sufficient it is enough to observe that equation 4 implies that there must exist a canonical coordinate η for T_a, and equation 8 that there exists a canonical coordinate ξ for S_b. These canonical coordinates can be found simultaneusly, since equations 4 and 8 hold for the same kernel, and this proves the assert.

It is well known that two operators have a common eigenfunction if and only if they commute (Davidov, 1965). It is easy to check that, for T_a, S_b, the family of such eigenfunctions is $k(\eta,\xi;u,v)$, the kernel of the integral transform written in the canonical coordinates η,ξ, and that the corresponding eigenvalues are $\exp(-iua), \exp(-ivb)$ respectively. In fact, $T_a k(\eta,\xi;u,v) = k(\eta+\delta\eta,\xi;u,v) = \exp(-iua)k(\eta,\xi;u,v)$, since in the coordinate system η,ξ the action of T_a is a translation along η. Analogusly, $S_b k(\eta,\xi;u,v) = k(\eta,\xi+\delta\xi;u,v) = \exp(-ivb)k(\eta,\xi;u,v)$. This amounts to say that given two transformation groups T_a, S_b there exists a coordinate systems in which the transformational states of a pattern have well defined values and can be measured simultaneously, and independently from the shape of the pattern. This is reminiscent of what happens in quantum mechanics, where operators associated with two physical quantities yield observables which can be measured simultaneously if and only if they commute. The reason for this analogy resides in the property of commutativy of T_a, S_b, in that it ensures that there is no "interaction" between the two transformations.

TWO-PARAMETERS TRANSFORMATION GROUPS

In this section the results concerning strong invariance will be extended to a particular class of *two-parameters transformation groups.* Preliminarily note that a the action of a two-parameter transformation group S_{ab} is essentially equivalent to the composition of the actions of two groups T_a, V_b (Bluman and Kumei, 1989). Then if the vector fields X_a, X_b, of T_a, V_b respectively, commute and are linerly independent there exists an integral transform invariant in the strong sense with respect to S_{ab}. For instance, it is straightforward to prove that a two-parameter transformation group such that x and y are transformed indepedently satisfies the condition for the existence of an integral transform invariant in the strong sense.

Proposition 3. *Suppose it is given a two parameter transformation group $S_{a,b}$, with parameters a and b, such that*

$$S_{ab}x = x'(x,a),$$
$$S_{ab}y = y'(y,b)$$

respectively. Then there exists an integral transform $\mathcal{G}$ invariant in the strong sense with respect to S_{ab}.

Proof. The action of S_{ab} on x, y can be written as the composition of two tranformations T_a, V_b such that $(T_a x, T_a y) = (x'(x,a), y)$ and $(V_b x, V_b y) = (x, y'(x,b))$. The vector fields of T_a, V_b are (Bauman and Kumei, 1989)

$$X_a = \frac{dx'}{da} |_{a=0} \frac{\partial}{\partial x},$$
$$X_b = \frac{dy'}{db} |_{b=0} \frac{\partial}{\partial y},$$

and it is immediate to prove that X_a, X_b commute since x' does not depend on y and y' is independent from x; furthermore X_a, X_b are orthogonal and hence linearly independent. Then the conditions for the existence of a representation invariant in the strong sense are met (Ferraro and Caelli, 1988).

Let η, ξ be the canonical coordinates of T_a, V_b respectively. By definition they must satisfy the following equations

$$X_a \eta = 1 \quad X_b \eta = 0,$$
$$X_a \xi = 0 \quad X_b \xi = 1, \tag{9}$$

(Ferraro and Caelli, 1988) that are two pairs of partial differential equations from which η and ξ can be computed as functions of x and y; the kernel of the integral transform invariant, in the strong sense, under $S_{a,b}$ is then

$k(x, y; u, v) =$

$$|J(\xi, \eta; x, y)| \exp\{-[u\eta(x,y) + v\xi(x,y)]\},$$

(see equation 3).

For instance consider the transformation $S_{ab}(x,y) = (\exp(a)x, \exp(b)y)$. (The reason for the use of this notation for the dilations is to ensure that the elements corresponding to $a = 0$, $b = 0$ belongs to the groups T_a, V_b respectively and indeed are the identities.) The corresponding vector fields are

$$X_a = x\partial/\partial x, \quad X_a = y\partial/\partial y.$$

Equations 9 then become

$$x\frac{\partial \eta}{\partial x} = 1, \quad x\frac{\partial \xi}{\partial x} = 0,$$

$$y\frac{\partial \eta}{\partial y} = 0, \quad y\frac{\partial \xi}{\partial y} = 1,$$

and one readily obtains the solutions $\eta(x,y) = \lg x$, $\xi(x,y) = \lg y$. The determinant of the Jacobian is $|J(\eta, \xi, x, y)| = |xy|^{-1}$; thus the kernel of the integral transform invariant with respect to S_{ab} is

$$k(x, y; u, v) = |xy|^{-1} \exp\{-[u \lg |x| + v \lg |y|]\},$$

that is $k(x, y; u, v) = x^{-u-1} y^{-v-1}$, the kernel of the two-dimensional Mellin transform (Caelli and Liu, 1988). It is easy to check that the under dilations the Mellin transform is indeed invariant in the strong sense (Ferraro, 1992).

For another instance of this procedure consider $S_{ab}(x,y) = (x^{\exp a}, y^{\exp b})$. In this case $X_a = x \lg x \partial/\partial x$, $X_b = y \lg y \partial/\partial y$, and

$$x \lg x \frac{\partial \eta}{\partial x} = 1, \quad x \lg x \frac{\partial \xi}{\partial x} = 0,$$

$$y \lg y \frac{\partial \eta}{\partial y} = 0, \quad y \lg y \frac{\partial \xi}{\partial y} = 1,$$

whose solutions are $\eta = \lg(\lg x)$, $\xi = \lg(\lg y)$; the Jacobian determinant is $|J(\eta, \xi; x, y)| = |xy \lg x \lg y|^{-1}$, and hence the kernel of the invariant integral transform is

$k(x, y; u, v) =$

$$|xy \lg x \lg y|^{-1} \exp\{-[u \lg(\lg x) + v \lg(\lg y)]\}.$$

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