SOME PROPERTIES AND ALGORITHMS FOR FOURTH ORDER SPECTRAL ANALYSIS OF COMPLEX SIGNALS

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ABSTRACT

Some algorithms for linear system identification based on fourth order spectra are given. They extend algorithms developed in the case of third order statistics. We also give a method for phase unwrapping for fourth order spectra and we establish a link between algorithms based on kurtosis maximization and identification method in the frequency domain.

1 INTRODUCTION

Fourth order statistics of complex signals are among the possible tools for treating the problem of channel identification and equalization in digital communication [9]. Parametric methods (in the time domain) have been studied for several years [4],[9]; it seems that the fourth order methods in the frequency domain have not been studied in depth. We know the works of Mendel [9] Dalle Molle and Hinich [2], Pan and Nikias [10] and Le Roux *et al.* [7],[6]. Pierce [12],[13] seems to be among the few who treated the case of complex signals with applications in the field of geophysics. Here, we extend a Fourier transform phase reconstruction algorithm that we developed in the case of third order spectra [5]. This multiresolution method does not raise phase unwrapping difficulties. If we intend to use optimal techniques [7],[6] phase unwrapping is necessary : the trispectrum phase is the sum of four spectrum phases (see eq. 3) and takes values in the interval $[-4\pi, 4\pi]$; the trispectrum phase is computed as the argument of the complex trispectrum and is wrapped in the interval $[-\pi, \pi]$. Some algorithms, especially those involving divisions, require a phase unwrapping step. Here we give a phase unwrapping method that extends the work of Marron *et al.* [8]. We apply this formula to our extension of the optimal reconstruction method [7],[6].

Our developments are based on the cumulant formula which never vanishes under circularity hypothesis [1][11]:

$$\begin{aligned} C_4^x(t_1,t_2,t_3) &= E\left\{x^*(\tau)x(\tau+t_1)x(\tau+t_2)x^*(\tau+t_3)\right\} \\ &-E\left\{x^*(\tau)x(\tau+t_1)\right\}E\left\{x(\tau+t_2)x^*(\tau+t_3)\right\} \\ &-E\left\{x(\tau+t_1)x(\tau+t_2)\right\}E\left\{x^*(\tau)x^*(\tau+t_3)\right\} \\ &-E\left\{x^*(\tau)x(\tau+t_2)\right\}E\left\{x(\tau+t_1)x^*(\tau+t_3)\right\}, \quad (1) \end{aligned}$$

and the corresponding fourth order spectrum [12][13] :

$$\begin{aligned} &T_4^X(\omega_1,\omega_2,\omega_3) = \quad (2) \\ &< X(\omega_1)X(\omega_2)X^*(-\omega_3)X^*(\omega_1+\omega_2+\omega_3) > \\ &- < X(\omega_1)X^*(\omega_1) >< X(\omega_2)X^*(\omega_2) > \delta(\omega_2+\omega_3) \\ &- < X(\omega_2)X^*(\omega_2) >< X(\omega_1)X^*(\omega_1) > \delta(\omega_1+\omega_3) \\ &- < X^*(\omega_3)X^*(-\omega_3) >< X(\omega_1)X(-\omega_1) > \delta(\omega_1+\omega_2). \end{aligned}$$

• Remark : We draw the attention of the reader on the importance of the three planes appearing in equation 2. If the analysed signal is the output of a LTI system driven by non-gaussien zero-mean IID sequence (Fig. 1), the fourth order spectrum phase satisfies:

$$\begin{aligned} \psi^Y(\omega_1,\omega_2,\omega_3) &= \hat{\varphi}^H(\omega_1) + \hat{\varphi}^H(\omega_2) \quad (3) \\ &-\hat{\varphi}^H(-\omega_3) - \hat{\varphi}^H(\omega_1+\omega_2+\omega_3) + k\pi, \end{aligned}$$

where $\psi^Y(\omega_1,\omega_2,\omega_3)$ is the trispectrum phase of the output, $\hat{\varphi}^H(\omega)$ is the system Fourier transform phase and $k = 0$ or 1 depending on the input kurtosis sign.

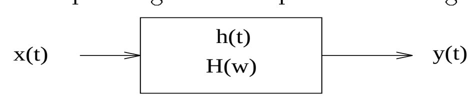

Figure 1: Identification scheme.

2 FOURIER PHASE RECONSTRUCTION FROM THE FOURTH ORDER SPECTRUM PHASE

2.1 A multiresolution reconstruction method

We give an extension of an algorithm developed in the third order case [5]:

$\hat{\varphi}^H(0) = \hat{\varphi}^H(1) = \hat{\varphi}^H(2) = 0$

(at the sampling frequency and its multiples)

For $n = 1, \ldots, \log_2 N$

$$\hat{\varphi}^H(\tfrac{1}{2^n}) = \tfrac{1}{2}\left[\begin{array}{l} \psi^Y(\frac{1}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) \\ \quad +\hat{\varphi}^H(\frac{-1}{2^{n-1}}) + \hat{\varphi}^H(\frac{1}{2^{n-2}}) \end{array}\right] \quad (4)$$

For $k = 0, \ldots, 2^{n-1}-2$

$$\begin{aligned} \hat{\varphi}^H(\tfrac{2k+3}{2^n}) &= -\psi^Y(\tfrac{k}{2^{n-1}},\tfrac{1}{2^n},\tfrac{1}{2^{n-1}}) + \hat{\varphi}^H(\tfrac{k}{2^{n-1}}) \\ &+\hat{\varphi}^H(\tfrac{1}{2^n}) - \hat{\varphi}^H(\tfrac{-1}{2^{n-1}}) \end{aligned} \quad (5)$$

Where $\hat{\varphi^H}(\omega)$ is the estimated Fourier phase and $\psi^Y(\omega 1, \omega 2, \omega 3)$ is the trispectrum phase.

- This algorithm requires no phase unwrapping.

2.2 Least square reconstruction

The criterion and the general formula for real signals are given in [7]. This reconstruction requires a prior phase unwrapping (cf. [8]). In the complex case, the optimal fourth order solution is obtained in minimizing:

$$\sum_{\omega_1=0}^{N}\sum_{\omega_2=0}^{N}\sum_{\omega_3=0}^{N} \|\psi_4^Y(\omega_1,\omega_2,\omega_3) - \hat{\psi}_4^H(\omega_1,\omega_2,\omega_3)\|^2. \quad (6)$$

The minimum is obtained when

$$\hat{\varphi}^H(\omega) = \frac{1}{2N^2}\sum_{\omega_1=0}^{N}\sum_{\omega_2=0}^{N}\psi_4^Y(\omega,\omega_1,\omega_2) - \frac{1}{2N^2}\sum_{\omega_1=0}^{N}\sum_{\omega_2=0}^{N}\psi_4^Y(\omega_1,\omega_2,-\omega) + K, \quad (7)$$

where K is an arbitrary constant.

2.2.1 Phase unwrapping algorithms

- *Marron's bispectrum phase unwrapping algorithm*

Marron *et al.* have shown that it is possible to deduce all the N^2 unwrapped bispectrum phases $\psi_3(\omega_1,\omega_2)$ from the $N-1$ modulo 2π bispectrum phases $\psi_3(1,\omega)$, $\omega = 1,2,\ldots,N-1$.
To obtain the unwrapped phases, they use the following equation:

$$\psi_3(\omega_1,\omega_2) = \psi_3(\omega_1-1,\omega_2+1) + \psi_3(1,\omega_2) - \psi_3(1,\omega_1-1). \quad (8)$$

In the next section, we extend the Marron's algorithm to the fourth order spectrum of complex signals. Moreover we deduce all the N^3 unwrapped trispectrum phases from the $(N-1)$ modulo 2π trispectrum phases used in the multiresolution algorithm(see eq. 4 and 5).

- *(α) Extension of Marron's algorithm for the fourth order spectrum*

The fourth order extension of Marron's formulas is represented in figure 2. Its expression is:

$$\psi(z+v+w,x,y) + \psi(w,v,z) = \psi(w,x,y) + \psi(w+x+y,v,z), \quad (9)$$

for all v,w,x,y,z.

Note that this formulas can be generalized to any order. It is always a four terms identity.

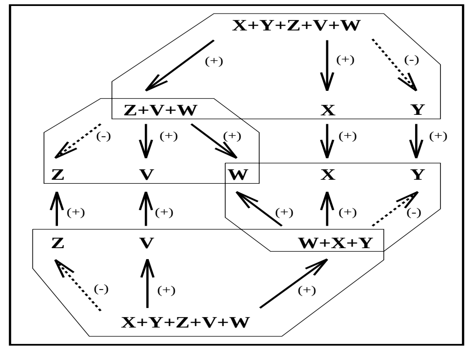

Figure 2: Scheme representing the relationship between four trispectrum phases. It illustrates eq.(9) when eq.(3) is satisfied.

The $(N-1)$ initial values are the trispectrum phases used in the multiresolution method (these do not require phase unwrapping). From this we deduce the N^3 unwrapped trispectrum phases by the following recursion.

For $n = 1,\ldots,\log_2 N$
For $u,v = 0,1,2\ldots,2^{n-1}-1$
For $x,y,z = 1,3,5,\ldots,2^n-1$

•(a) $\psi(\frac{2u+3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) = \psi(\frac{u}{2^{n-1}},\frac{1}{2^{n-2}},\frac{1}{2^{n-1}}) + \psi(\frac{1}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) - \psi(\frac{u}{2^{n-1}},\frac{1}{2^n},\frac{1}{2^{n-1}})$

•(b) $\psi(\frac{x}{2^n},\frac{y}{2^n},\frac{1}{2^{n-1}}) = \psi(\frac{y}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) + \psi(\frac{y+3}{2^n},\frac{x-3}{2^n},\frac{1}{2^{n-1}}) - \psi(\frac{x-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}})$

•(c) $\psi(\frac{u}{2^{n-1}},\frac{x}{2^n},\frac{1}{2^{n-1}}) = \psi(\frac{2u-3}{2^n},\frac{x}{2^n},\frac{1}{2^{n-1}}) + \psi(\frac{2u+x-1}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) - \psi(\frac{2u-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}})$

•(d) $\psi(\frac{x}{2^n},\frac{y}{2^{n-1}},\frac{u}{2^{n-1}}) = \psi(\frac{x-3}{2^n},\frac{y}{2^{n-1}},\frac{u}{2^{n-1}}) + \psi(\frac{x+2y+2u-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) - \psi(\frac{x-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}})$

•(e) $\psi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}) = \psi(\frac{-y-z-2}{2^n},\frac{y}{2^n},\frac{1}{2^{n-1}}) + \psi(\frac{-1}{2^{n-1}},\frac{x+y+z-2}{2^n},\frac{1}{2^{n-2}}) - \psi(\frac{-y-z-2}{2^n},\frac{x+y+z-2}{2^n},\frac{1}{2^{n-2}})$

•(f) $\psi(\frac{u}{2^{n-1}},\frac{x}{2^n},\frac{y}{2^n}) = \psi(\frac{-x-y-2}{2^n},\frac{x}{2^n},\frac{1}{2^{n-1}}) + \psi(\frac{-1}{2^{n-1}},\frac{2u+x+y-2}{2^n},\frac{1}{2^{n-2}}) - \psi(\frac{-x-y-2}{2^n},\frac{2u+x+y-2}{2^n},\frac{1}{2^{n-2}})$

•(g) $\psi(\frac{x}{2^n},\frac{y}{2^n},\frac{u}{2^{n-1}}) = \psi(\frac{-y-2u-2}{2^n},\frac{y}{2^n},\frac{1}{2^{n-1}}) + \psi(\frac{-1}{2^{n-1}},\frac{x+y+2u-2}{2^n},\frac{1}{2^{n-2}}) - \psi(\frac{-y-2u-2}{2^n},\frac{x+y+2u-2}{2^n},\frac{1}{2^{n-2}})$

•(h) $\psi(\frac{u}{2^{n-1}},\frac{v}{2^{n-1}},\frac{x}{2^n}) = \psi(\frac{2u-3}{2^n},\frac{v}{2^{n-1}},\frac{x}{2^n}) + \psi(\frac{2u+2v+x-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}}) - \psi(\frac{2u-3}{2^n},\frac{1}{2^n},\frac{1}{2^{n-1}})$

The trispectrum phases deduced by this unwrapping procedure is compared to the measured modulo 2π trispectrum phase and a phase of $2k\pi$ is added to the measured phase so that their difference will be less than π. The algorithm has the same iterative structure as

the multiresolution algorithm.
There are other approaches for performing phase unwrapping. Here are two methods that are also efficient in practice.

• *(β) Multiresolution used to unwrap the trispectrum phase*
The multiresolution method can be used to give a first approximation of the channel phases $\hat{\varphi}^H(\omega)$. Those values are then used to calculate the trispectrum phase in the interval $[-4\pi, 4\pi]$ using equation (3). A better solution consists in combining the multiresolution method with the optimal method as shown in the next section.

• *(γ) Multiresolution combined with the optimal method*
Such a combination is possible thanks to the iterative structure of the multiresolution method. The multiresolution will be combined with the optimal method as follows :

- at the step n of the algorithm, the multiresolution method gives a first estimate of $\hat{\varphi}^H\left(\frac{2m+1}{2^n}\right)$ for $m = 0, 1, ..., 2^{n-1} - 1$.
- these values, and those calculated in the previous steps (at lower resolutions), are used to unwrap the trispectrum phases : $\psi^Y\left(\frac{p}{2^n}, \frac{q}{2^n}, \frac{r}{2^n}\right)$ for $p, q, r = 0, 1, ..., 2^n - 1$ using equation (3).
- Next, the LS estimation (eq.(7)) uses these trispectrum phases to give an improved estimation of $\hat{\varphi}^H\left(\frac{2m+1}{2^n}\right)$ for $m = 0, 1, ..., 2^{n-1} - 1$.
- Finally, these last estimates will be used to initialize the next step $(n + 1)$ of the algorithm.

3 IDENTIFICATION AND KURTOSIS MAXIMIZATION

In this section, we show that the optimal least-squares identification is very similar to the well known kurtosis maximization.

3.1 Kurtosis maximization criterion

In this section, we use the scheme used in the equalization context:

The criterion was proposed by D. Donoho and later

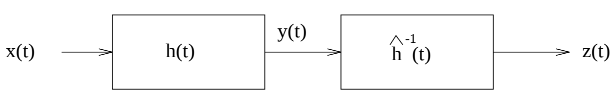

Figure 3: Equalization scheme.

by O. Shalvi and E. Weinstein [3] [14] [16] in order to identify the system $h(t)$. It consists in estimating $\hat{h}^{-1}(t)$ through the maximization of $K(z)$ under the constraint $E\left\{|z|^2\right\} = E\left\{|x|^2\right\}$ where $K(z)$ is the kurtosis of z.

In a first step O. Shalvi and E. Weinstein propose to whiten the output signal so that they are essentially reconstructing the channel Fourier transform phase just like the reconstruction algorithms in the frequency domain.

3.1.1 Expression of the kurtosis maximization in the frequency domain

$$K(z) = C_4^z(0,0,0) = \sum_{\omega_1=0}^{N} \sum_{\omega_2=0}^{N} \sum_{\omega_3=0}^{N} T^Z(\omega_1, \omega_2, \omega_3)$$
$$= \sum_{\omega_1=0}^{N} \sum_{\omega_2=0}^{N} \sum_{\omega_3=0}^{N} \left|T^Z(\omega_1, \omega_2, \omega_3)\right| e^{j\psi^Z(\omega_1,\omega_2,\omega_3)}. \quad (10)$$

• If the output is whitened, $\left|T^Z\right|$ is constant.
• If the output kurtosis is negative, then the input kurtosis is also negative and we replace ψ^Y by $(\psi^Y - \pi)$
• The kurtosis is a real number and the criterion is also real so we look for the phase maximizing

$$J = \frac{K(z)}{|T^Z|} = \quad (11)$$
$$\sum_{\omega_1=0}^{N-1} \sum_{\omega_2=0}^{N-1} \sum_{\omega_3=0}^{N-1} \cos\left[\psi^Y(\omega_1, \omega_2, \omega_3) - \hat{\psi}^H(\omega_1, \omega_2, \omega_3)\right].$$

3.1.2 Taylor expansion of the kurtosis

If the model is correct and his trispectrum phase is accurately estimated, the difference between $\psi^Y(\omega_1, \omega_2, \omega_3)$ and $\hat{\psi}^H(\omega_1, \omega_2, \omega_3)$ will be small and we can expand eq. (11) :

$$J = \sum_{\omega_1=0}^{N} \sum_{\omega_2=0}^{N} \sum_{\omega_3=0}^{N} \left[1 - \frac{1}{2}\left[\psi^Y(\omega_1, \omega_2, \omega_3) - \hat{\psi}^H(\omega_1, \omega_2, \omega_3)\right]^2 \right. \quad (12)$$
$$\left. + \frac{1}{24}\left[\psi^Y(\omega_1, \omega_2, \omega_3) - \hat{\psi}^H(\omega_1, \omega_2, \omega_3)\right]^4 + \ldots\right].$$

If we limit the development to the second term, the maximization of this criterion reduces to the minimization of the LS criterion obtained in the frequency domain (eq. 6). Under this hypothesis, kurtosis maximization reduces to the minimization of a quadratic criterion and the non-convexity difficulties are replaced by a phase unwrapping problem which has a solution if the model is correct and the trispectrum phase estimate accurate enough.

4 SIMULATION RESULTS

We have simulated a channel using the 26th order complex FIR filter proposed in [15] and deduced from experimental data. The input was a 4-QAM IID signal. Note that the input kurtosis is negative.

We have computed the trispectrum by

$$T(\omega_1,\omega_2,\omega_3)=\frac{1}{M}\sum_{i=1}^{M}\left[Y_n(\omega_1)Y_n(\omega_2)\right. \times Y_n^*(-\omega_3)Y_n^*(\omega_1+\omega_2+\omega_3)\left.\right], \quad (13)$$

where the $Y_n(\omega)$ are the Fourier transforms of a 64 samples long sequences of $y(t)$.

Figure 4 shows the analysed channel frequency response modulus. Figure 5 shows the results of the multiresolution method alone (trispectrum averages on 10000 and 50000 sequences). Figure 6 shows the results of the optimal method using the extension of Marron (α) and the combination (γ) unwrapping methods (trispectrum averages on 10000 sequences). The phase unwrapping methods α and β give comparable results while γ improves slightly the results.

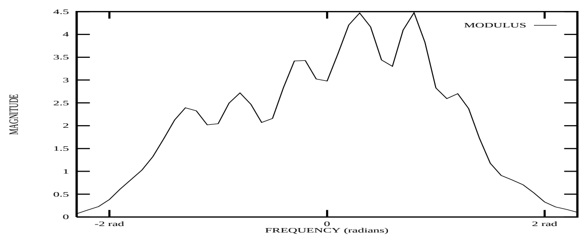

Figure 4: Fourier transform modulus : $|H(\omega)|$.

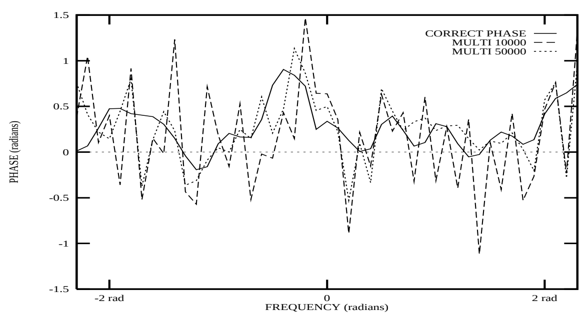

Figure 5: Fourier transform phase reconstruction with the multiresolution after 10000 (dashed line) and 50000 (dotted line) iterations .

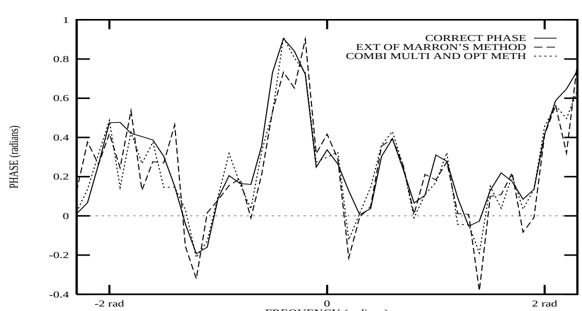

Figure 6: Fourier transform phase reconstruction after 10000 iterations.

References

[1] **P. COMON (1994):** "Circularité et signaux aléatoires à temps discrets," *Traitement du signal*, Vol 11 num 5, pages 417–420, 1994.

[2] **J.W. DALLE MOLLE and M.J. HINICH (1989):** "The Trispectrum," *IEEE Signal Processing ATHOS Workshop on HOS*, pages 68–72, Vail, Colorado, June 1989.

[3] **D. DONOHO (1981):** "On Minimum Entropy Deconvolution," *Applied Time Series Analysis II,D.F.Fiendley ed. Academic Press, New York*, pages 565–608, 1981.

[4] **G. GIANNAKIS and J.M. MENDEL (1989):** "Identification of non-minimum phase systems using Higher order statistics," *IEEE Trans. On ASSP*, Vol 37(3), pages 360–377, March 1989.

[5] **J. LE ROUX, D. ROSSILLE and C.HUET(1995):** "A multiresolution extension of Lohman Weigelt and Wirnitzer recursion for computing a Fourier transform phase from the third order spectrum phase," *IEEE Signal Processing ATHOS Workshop on HOS*, pages 315–319, Begur, Spain, June 1995.

[6] **J. LE ROUX and P. SOLE (1994):** "Least-squared error reconstruction of a sampled signal Fourier transform from its n-th order polyspectrum," *Signal Processing*, Vol 35, pages 75–81.

[7] **J. LE ROUX, P. SOLE, A.M. TEKALP and A.T. ERDEM (1993):** "Tekalp-Erdem estimator gives the LS estimate for Fourier phase and log-Fourier phase modulus," *IEEE Trans. On Signal Processing*, Vol 41(4), pages 1705–1707, April 1993.

[8] **J.C. MARRON , P.P. SANCHEZ and R.C. SULLIVAN (1990):** "Unwrapping algorithm for least-squares phase recovery from the modulo 2π bispectrum phase," *J.Opt.Soc.Am. A/Vol.7*, pages 14–20, January 1990.

[9] **J.M. MENDEL (1991):** "Tutorial on Higher-Order Statistics (Spectra) in Signal Processing and System Theory : Theoretical Results and Some Applications," *Proceedings of the IEEE*, Vol 79(3), pages 278–305, March 1991.

[10] **R. PAN and C.L. NIKIAS (1987):** "Phase Reconstruction in the Trispectrum Domain," *IEEE Trans. On Signal Processing*, Vol ASSP-35(6), pages 895–897, June 1987.

[11] **B. PICINBONO (1994):** "On circularity," *IEEE Trans. On Signal Processing*, Vol 42(12), pages 3473–3482, December 1994.

[12] **R. PIERCE (1991):** "Application of Higher Order Spectra to High Resolution Radar Measurement," *IEEE Signal Processing ATHOS Workshop on HOS*, pages 207–210, Chamrousse, France, July 1991.

[13] **R. PIERCE (1993):** "Velocity Measurement of Radar Target using HOS," *IEEE Signal Processing ATHOS Workshop on HOS*, pages 164–167, Lake Tahoe, June 1993.

[14] **O. SHALVI and E. WEINSTEIN (1990):** "New criteria for blind deconvolution of nonminimum phase systems (channels)," *IEEE Trans. On Information theory* , Vol 36(2), pages 312–321, March 1990.

[15] **J.J. SHYNK, R.P. GOOCH, G. KRISHNAMURTHY and C.K. CHAN(1991):** "A comparative performance study of several blind equalization algorithms," *In Proc. SPIE 1992 Adaptative Signal Processing*, Vol. 1565, pages 102–117, Jan. 1991.

[16] **J. TUGNAIT (1992):** "Comments on "New criteria for blind deconvolution of nonminimum phase systems (channels)"," *IEEE Trans. On Information theory* , Vol 38(1), pages 210–213, January 1992.