LINEAR TIME-VARIANT PROCESSING OF HIGHER-ORDER ALMOST-PERIODICALLY CORRELATED TIME-SERIES

Luciano Izzo *Antonio Napolitano*

Università di Napoli Federico II, Dipartimento di Ingegneria Elettronica
via Claudio 21, I-80125 Napoli, Italy; Tel: +39-81-7683156; Fax: +39-81-7683149
E-mail: izzo@nadis.dis.unina.it

ABSTRACT

The characterization and linear time-variant processing of the higher-order almost-periodically correlated time-series in the fraction-of-time probability framework are considered. At first, the characterization in the temporal domain is presented by exploiting the expression of the temporal moment function as a sum of complex sinusoids whose amplitudes and frequencies are continuous functions of the lag vector. Then, the characterization in the frequency domain is considered. Finally, for both random and nonrandom linear systems, the input/output relationships in terms of generalized cyclic temporal moment functions and generalized cyclic spectral moment functions are stated. As special cases, linear almost-periodically time-variant systems as well as systems performing time-scale changing are also treated.

1 INTRODUCTION

In the last years, the theory of second-order wide-sense cyclostationarity has begun to be generalized to a theory of signals exhibiting higher-order wide-sense cyclostationarity (WSCS) [1]-[5]. For such signals there exist higher- than second-order time-invariant transformations that convert into spectral lines (whose frequencies are called cycle frequencies) the hidden periodicities due to some operations, such as modulation, sampling, coding, and multiplexing. Signals exhibiting WSCS are said to be almost cyclostationary when the set of all cycle frequencies is countable. They can be characterized in the temporal domain in terms of cyclic temporal moment and cyclic temporal cumulant functions, which are the Fourier coefficients of the temporal moment function and the temporal cumulant function, respectively. Moreover, since the cyclic temporal moment functions are continuous functions of the lag vector in the origin, such signals can also be characterized in the frequency domain in terms of cyclic spectral moment functions and cyclic polyspectra [1].

A wider class of nonstationary signals is that of the higher-order almost-periodically correlated (APC) signals for which the cyclic temporal moment functions are not necessarily continuous functions of the lag vector in the origin. In such a case, the set of cycle frequencies is not necessarily countable and a spectral characterization in terms of Fourier transforms of the cyclic temporal moment functions is not easy. The class of APC signals includes, as a special case, that of the almost-cyclostationary signals. Moreover, chirp and polynomial modulations as well as exponential modulation and some linear time-variant transformations of almost-cyclostationary signals give rise to APC signals. It is worthwhile to point out that the adopted definition of APC signals is in agreement with that given in [6], [7] with reference to second-order statistics. Moreover, note that in [6], [7], the entire theory is practically limited to APC signals with second-order cyclic temporal moment functions that are continuous functions of the lag parameter in the origin (almost-cyclostationary signals).

The present paper deals with the characterization and linear processing of the higher-order APC signals. Specifically, with reference to the fraction-of-time (FOT) probability framework, it is shown that the temporal moment function can be expressed as a sum of complex sinusoids whose amplitudes and frequencies are continuous functions of the lag vector. Then, starting from such a representation, the characterization in terms of generalized cyclic temporal moment functions and generalized cyclic spectral moment functions is presented. Subsequently, the way in which the higher-order statistics of APC time-series change as they are processed by random and nonrandom (in the FOT probability sense) linear time-variant systems is investigated. Nonrandom systems are those that for every deterministic (i.e., constant, periodic, or polyperiodic) input time-series deliver a deterministic output time-series. They include the linear almost-periodically time-variant (LAPTV) systems as well as the systems that perform a time-scale changing [4]. Random systems are all the time-variant transformations that cannot be modeled as nonrandom. They include chirp modulators, modulators whose carrier is a pseudo-noise sequence (as in the spread spectrum modulation), channels introducing time-varying delays, and systems that perform a time windowing [5].

2 N-TH ORDER ALMOST-PERIODICALLY CORRELATED TIME-SERIES

In the FOT probability context, a continuous-time possibly complex-valued time-series $x(t)$ is said to exhibit Nth-order wide-sense cyclostationarity with cycle frequency $\alpha \neq 0$, for a given conjugation configuration, if the Nth-order cyclic temporal moment function (CTMF)

$$\mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N \triangleq \left\langle \prod_{n=1}^{N} x^{(*)_n}(t+\tau_n) e^{-j2\pi\alpha t} \right\rangle \qquad (1)$$

is not zero for some $\boldsymbol{\tau}$ [1]. In (1), $\boldsymbol{\tau} \triangleq [\tau_1, ..., \tau_N]^{\mathrm{T}}$ and $\boldsymbol{x} \triangleq [x^{(*)_1}(t), ..., x^{(*)_N}(t)]^{\mathrm{T}}$ are column vectors, $\langle\cdot\rangle$ denotes infinite time averaging, and $(*)_n$ represents optional conjugation. If the set

$$A_{\boldsymbol{\tau}} \triangleq \{\alpha \in \mathsf{R} \; : \; \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N \neq 0\} \qquad (2)$$

is countable for each $\boldsymbol{\tau}$, the time-series is said to be almost-periodically correlated (for the considered conjugation configuration) [6], [7], and the almost-periodic function defined as

$$\mathcal{R}_{\boldsymbol{x}}(t, \boldsymbol{\tau})_N \triangleq \sum_{\alpha \in A_{\boldsymbol{\tau}}} \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N e^{j2\pi\alpha t} \qquad (3)$$

is referred to as the temporal moment function.

In the case where the set

$$A \triangleq \bigcup_{\boldsymbol{\tau} \in \mathsf{R}^N} A_{\boldsymbol{\tau}} \qquad (4)$$

is countable, the time-series $x(t)$ is said to be wide-sense almost-cyclostationary and, moreover, the function $\mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N$ turns out to be continuous with respect to $\boldsymbol{\tau}$ [1].

Almost-cyclostationary time-series can be characterized in the frequency domain by the N-fold Fourier transform $\mathcal{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f})_N$ of the CTMF, which is called the Nth-order cyclic spectral moment function (CSMF) and can be written as [1]

$$\mathcal{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f})_N = S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')_N \delta(\boldsymbol{f}^{\mathrm{T}}\mathbf{1} - \alpha), \qquad (5)$$

where $\delta(\cdot)$ is Dirac's delta function, $\boldsymbol{f} \triangleq [f_1, ..., f_N]^{\mathrm{T}}$, $\mathbf{1} \triangleq [1, ..., 1]^{\mathrm{T}}$, and prime denotes the operator that transforms a vector $\boldsymbol{w} \triangleq [w_1, ..., w_N]^{\mathrm{T}}$ into $\boldsymbol{w}' \triangleq [w_1, ..., w_{N-1}]^{\mathrm{T}}$. The function $S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')_N$, referred to as the Nth-order reduced-dimension CSMF (RD-CSMF), can be expressed as the $(N-1)$-fold Fourier transform of the Nth-order reduced-dimension CTMF defined by setting $\tau_N = 0$ into (1). Such a characterization is not appropriate for those APC time-series that are not almost-cyclostationary since the lack of continuity of the CTMFs can lead to infinitesimal RD-CSMFs.

A useful characterization of the APC time-series in the frequency domain can be introduced under the assumption that the set $A_{\boldsymbol{\tau}}$ is continuous with respect to $\boldsymbol{\tau}$. In such a case, it can be shown that the temporal moment function (3) can be written as

$$\mathcal{R}_{\boldsymbol{x}}(t, \boldsymbol{\tau})_N = \sum_{\zeta \in W} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N e^{j2\pi\alpha_\zeta(\boldsymbol{\tau})t}, \qquad (6)$$

where W denotes a countable set, the lag-dependent cycle frequencies $\alpha_\zeta(\boldsymbol{\tau})$ are continuous functions of $\boldsymbol{\tau}$ defining the support in the $(\alpha, \boldsymbol{\tau})$ space of the CTMF, that is,

$$\operatorname{supp}\{\mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N\} = \left\{(\alpha, \boldsymbol{\tau}) \in A_{\boldsymbol{\tau}} \times \mathsf{R}^N \; : \; \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N \neq 0\right\}$$

$$= \bigcup_{\zeta \in W} \left\{(\alpha, \boldsymbol{\tau}) \in \mathsf{R} \times \mathsf{R}^N \; : \; \alpha = \alpha_\zeta(\boldsymbol{\tau}), \; \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N \neq 0\right\}, \qquad (7)$$

and the functions

$$\mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N \triangleq \lim_{\Delta\boldsymbol{\tau} \to \mathbf{0}} \left\langle \prod_{n=1}^{N} x^{(*)_n}(t+\tau_n+\Delta\tau_n) e^{-j2\pi\alpha_\zeta(\boldsymbol{\tau}+\Delta\boldsymbol{\tau})t} \right\rangle, \qquad (8)$$

referred to as the generalized CTMFs (GCTMFs), are continuous functions also when the CTMFs are not. It is useful to point out that the limit operation is introduced into definition (8) to avoid discontinuities in $\mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N$ in correspondence of those $\boldsymbol{\tau}$ such that, for some $\zeta' \neq \zeta$, $\alpha_{\zeta'}(\boldsymbol{\tau}) = \alpha_\zeta(\boldsymbol{\tau})$.

It can be easily shown that CTMFs and GCTMFs are related by the following relationships:

$$\mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N = \lim_{\Delta\boldsymbol{\tau} \to \mathbf{0}} \mathcal{R}_{\boldsymbol{x}}^{\alpha_\zeta(\boldsymbol{\tau}+\Delta\boldsymbol{\tau})}(\boldsymbol{\tau}+\Delta\boldsymbol{\tau})_N, \qquad (9)$$

$$\mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N = \sum_{\zeta \in W} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N \delta_{\alpha - \alpha_\zeta(\boldsymbol{\tau})}, \qquad (10)$$

where $\delta_\gamma = 1$ for $\gamma = 0$ and $\delta_\gamma = 0$ for $\gamma \neq 0$.

Let us note that for the almost-cyclostationary time-series the functions $\alpha_\zeta(\boldsymbol{\tau})$ are independent of $\boldsymbol{\tau}$ and then there exists a one-to-one correspondence between the elements ζ of the set W and the cycle frequencies α belonging to the set A. Moreover, for each α and ζ such that $\alpha_\zeta(\boldsymbol{\tau}) = \alpha$, it results that

$$\mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N = \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})_N. \qquad (11)$$

The N-fold Fourier transform of the GCTMF

$$\mathcal{S}_{\boldsymbol{x},\zeta}(\boldsymbol{f})_N \triangleq \int_{\mathsf{R}^N} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N e^{-j2\pi\boldsymbol{f}^{\mathrm{T}}\boldsymbol{\tau}} \mathrm{d}\boldsymbol{\tau} \qquad (12)$$

is called the Nth-order generalized CSMF (GCSMF). It can be expressed as

$$\mathcal{S}_{\boldsymbol{x},\zeta}(\boldsymbol{f})_N =$$

$$\int_{\mathbb{R}^{N-1}} R_{\boldsymbol{x},\zeta}(\boldsymbol{\tau}')_N \delta(\alpha_\zeta(\boldsymbol{\tau}') - \boldsymbol{f}^{\mathrm{T}}\mathbf{1}) e^{-j2\pi \boldsymbol{f}'^{\mathrm{T}}\boldsymbol{\tau}'} \, \mathrm{d}\boldsymbol{\tau}', \quad (13)$$

where

$$R_{\boldsymbol{x},\zeta}(\boldsymbol{\tau}')_N \triangleq \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N \Big|_{\tau_N=0} \quad (14)$$

is the reduced-dimension GCTMF.

Let us note that, accounting for (11), for the almost-cyclostationary time-series the GCSMFs are coincident with the CSMFs.

3 EFFECTS OF LINEAR SYSTEMS ON APC TIME-SERIES

3.1 Random and Nonrandom Linear Systems

In the FOT probability framework, random and nonrandom systems are possibly complex (and not necessarily linear) systems that, for every deterministic (i.e., constant, periodic or polyperiodic) input time-series, deliver a nondeterministic or deterministic output time-series, respectively [4], [5].

With reference to the class of linear systems, a random system can be characterized by the transmission function

$$\mathcal{H}(f,\lambda) = \int_{-\infty}^{+\infty} H(s,f)\, \delta(\lambda - \psi(s,f))\, \mathrm{d}s, \quad (15)$$

where $H(s,f)$ and $\psi(s,f)$ are, for each s, a complex function and a monotonic (with respect to f) real function, respectively [5].

By inverse Fourier transforming both sides of (15), one obtains the expression of the impulse-response function

$$\mathfrak{h}(t,u) = \int_{-\infty}^{+\infty} h(s,t) \underset{t}{\otimes} \Psi(s,t,u)\, \mathrm{d}s, \quad (16)$$

where $\underset{t}{\otimes}$ denotes convolution with respect to t, $h(s,t)$ is the inverse Fourier transform of $H(s,f)$, and

$$\Psi(s,t,u) \triangleq \int_{-\infty}^{+\infty} e^{-j2\pi\psi(s,f)u} e^{j2\pi f t}\, \mathrm{d}f. \quad (17)$$

The nonrandom systems can be viewed as special cases of the random ones where

$$H(s,f) = \sum_{\sigma\in\Omega} H_\sigma(f) \delta(s-\sigma) \quad (18)$$

and

$$\psi(s,f) = \begin{cases} \psi_\sigma(f) & s=\sigma,\ \sigma\in\Omega, \\ \text{undetermined} & \text{elsewhere}, \end{cases} \quad (19)$$

where Ω is a countable set. Therefore, for nonrandom systems, the transmission function and the impulse-response function can be written as [4]

$$\mathcal{H}(f,\lambda) = \sum_{\sigma\in\Omega} H_\sigma(f)\, \delta(\lambda - \psi_\sigma(f)) \quad (20)$$

and

$$\mathfrak{h}(t,u) = \sum_{\sigma\in\Omega} h_\sigma(t) \underset{t}{\otimes} \Psi_\sigma(t,u), \quad (21)$$

where $h_\sigma(t)$ is the inverse Fourier transform of $H_\sigma(f)$ and

$$\Psi_\sigma(t,u) \triangleq \Psi(s,t,u)\big|_{s=\sigma}, \ \sigma\in\Omega. \quad (22)$$

3.2 Input/Output Relations for Linear Systems

Let us consider a random linear system excited by an APC time-series $x(t)$ whose set of Nth-order lag-dependent cycle frequencies, for the considered conjugation configuration, is $\{\alpha_\zeta(\boldsymbol{\tau})\}_{\zeta\in W_x}$.

The Nth-order CTMF at the cycle frequency β of the output time-series $y(t)$ can be derived accounting for (1) and (16) :

$$\mathcal{R}_{\boldsymbol{y}}^{\beta}(\boldsymbol{\tau})_N = \int_{\mathbb{R}^N} \left(\prod_{n=1}^{N} h^{(*)_n}(s_n,\tau_n) \right)$$

$$\underset{\boldsymbol{\tau}}{\otimes} \sum_{\zeta\in W_x} \int_{\mathbb{R}^N} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{v})_N \mathcal{R}_{\boldsymbol{\Psi}(\boldsymbol{s})}^{\beta}(\boldsymbol{\tau},\boldsymbol{v})_N \, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{s}, \quad (23)$$

where $\underset{\boldsymbol{\tau}}{\otimes}$ denotes N-dimensional convolution with respect to $\boldsymbol{\tau}$ and

$$\mathcal{R}_{\boldsymbol{\Psi}(\boldsymbol{s})}^{\beta}(\boldsymbol{\tau},\boldsymbol{v})_N \triangleq \left\langle \prod_{n=1}^{N} \Psi^{(*)_n}(s_n, t+\tau_n, v_n) e^{-j2\pi\beta t} \right\rangle. \quad (24)$$

Note that, in the particular case of nonrandom systems, (23) reduces to

$$\mathcal{R}_{\boldsymbol{y}}^{\beta}(\boldsymbol{\tau})_N = \sum_{\boldsymbol{\sigma}\in\Omega^N} \left(\prod_{n=1}^{N} h_{\sigma_n}^{(*)_n}(\tau_n) \right)$$

$$\underset{\boldsymbol{\tau}}{\otimes} \sum_{\zeta\in W_x} \int_{\mathbb{R}^N} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{v})_N \mathcal{R}_{\boldsymbol{\Psi}_{\boldsymbol{\sigma}}}^{\beta}(\boldsymbol{\tau},\boldsymbol{v})_N \, \mathrm{d}\boldsymbol{v}, \quad (25)$$

where $\mathcal{R}_{\boldsymbol{\Psi}_{\boldsymbol{\sigma}}}^{\beta}(\boldsymbol{\tau},\boldsymbol{v})_N$ is defined according to (22) and (24).

When the set of output cycle frequencies for each fixed value of $\boldsymbol{\tau}$ is countable and the set $\{\beta_\eta(\boldsymbol{\tau})\}_{\eta\in W_y}$ of potential output lag-dependent cycle frequencies is known, accounting for (9), the input/output relationship in terms of GCTMFs can be derived from (23):

$$\mathcal{R}_{\boldsymbol{y},\eta}(\boldsymbol{\tau})_N = \int_{\mathbb{R}^N} \left(\prod_{n=1}^{N} h^{(*)_n}(s_n,\tau_n) \right)$$

$$\underset{\boldsymbol{\tau}}{\otimes} \sum_{\zeta\in W_x} \int_{\mathbb{R}^N} \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{v})_N \overline{\mathcal{R}}_{\boldsymbol{\Psi}(\boldsymbol{s})}^{\beta_\eta(\boldsymbol{\tau})}(\boldsymbol{\tau},\boldsymbol{v})_N \, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{s}, \quad (26)$$

where

$$\overline{\mathcal{R}}_{\boldsymbol{\Psi}(\boldsymbol{s})}^{\beta_\eta(\boldsymbol{\tau})}(\boldsymbol{\tau},\boldsymbol{v})_N \triangleq \lim_{\Delta\boldsymbol{\tau}\to 0} \mathcal{R}_{\boldsymbol{\Psi}(\boldsymbol{s})}^{\beta_\eta(\boldsymbol{\tau}+\Delta\boldsymbol{\tau})}(\boldsymbol{\tau}+\Delta\boldsymbol{\tau},\boldsymbol{v})_N. \quad (27)$$

Moreover, taking the N-fold Fourier transform of both sides of (26), one obtains the input/output relationship in terms of GCSMFs:

$$\mathcal{S}_{\boldsymbol{y},\eta}(\boldsymbol{f})_N = \sum_{\zeta \in W_x} \int_{\mathbb{R}^N} \left(\prod_{n=1}^{N} H^{(*)_n}(s_n, {}_{(-)_n} f_n) \right)$$

$$\cdot \int_{\mathbb{R}^N} \mathcal{S}_{\boldsymbol{x},\zeta}\left(\boldsymbol{\psi}^{(-)}(\boldsymbol{s}, \boldsymbol{\lambda}^{(-)})\right)_N \Delta(\beta_\eta(\cdot); \boldsymbol{f}, \boldsymbol{\lambda}) \, \mathrm{d}\boldsymbol{\lambda} \, \mathrm{d}\boldsymbol{s}, \quad (28)$$

where $(-)_n$ denotes an optional minus sign to be considered only when the optional conjugation $(*)_n$ is present, $\boldsymbol{\lambda}^{(-)} \triangleq [{}_{(-)_1}\lambda_1, \ldots, {}_{(-)_N}\lambda_N]^{\mathrm{T}}$, $\boldsymbol{\psi}^{(-)}(\boldsymbol{s}, \boldsymbol{\lambda}) \triangleq [{}_{(-)_1}\psi(s_1, \lambda_1), \ldots, {}_{(-)_N}\psi(s_N, \lambda_N)]^{\mathrm{T}}$, and

$$\Delta(\beta_\eta(\cdot); \boldsymbol{f}, \boldsymbol{\lambda}) \triangleq$$

$$\lim_{\Delta\boldsymbol{\tau} \to \boldsymbol{0}} \int_{\mathbb{R}^N} \delta_{\beta_\eta(\boldsymbol{\tau}+\Delta\boldsymbol{\tau}) - \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{1}} \, e^{-j2\pi(\boldsymbol{f}-\boldsymbol{\lambda})^{\mathrm{T}}\boldsymbol{\tau}} e^{-j2\pi\boldsymbol{\lambda}^{\mathrm{T}}\Delta\boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{\tau}. \quad (29)$$

Let us note that in general the determination of the set $\{\beta_\eta(\boldsymbol{\tau})\}_{\eta \in W_y}$ is not straightforward. However, for LAPTV systems and systems performing a time-scale changing such a set can be easily singled out. Specifically, since for LAPTV systems the impulse-response function is given by

$$\mathfrak{h}(t, u) = \sum_{\sigma \in \Omega} h_\sigma(t-u) e^{j2\pi\sigma u}, \quad (30)$$

accounting for (25), it results that

$$\beta_\eta(\boldsymbol{\tau}) = \alpha_\xi(\boldsymbol{\tau}) + \boldsymbol{\rho}^{(-)\mathrm{T}}\boldsymbol{1}, \quad (31)$$

where $\xi \in W_x$ and $\boldsymbol{\rho} \in \Omega^N$, and hence (26) reduces to

$$\mathcal{R}_{\boldsymbol{y},\eta}(\boldsymbol{\tau})_N =$$

$$\sum_{\zeta \in W_x} \sum_{\boldsymbol{\sigma} \in \Omega^N} \left(\prod_{n=1}^{N} h_{\sigma_n}^{(*)_n}(\tau_n) \right) \underset{\boldsymbol{\tau}}{\otimes} \left\{ \mathcal{R}_{\boldsymbol{x},\zeta}(\boldsymbol{\tau})_N e^{j2\pi\boldsymbol{\sigma}^{(-)\mathrm{T}}\boldsymbol{\tau}} \right.$$

$$\left. \lim_{\Delta\boldsymbol{\tau} \to \boldsymbol{0}} \delta_{\alpha_\zeta(\boldsymbol{\tau}+\Delta\boldsymbol{\tau}) + \boldsymbol{\sigma}^{(-)\mathrm{T}}\boldsymbol{1} - \alpha_\xi(\boldsymbol{\tau}+\Delta\boldsymbol{\tau}) - \boldsymbol{\rho}^{(-)\mathrm{T}}\boldsymbol{1}} \right\}. \quad (32)$$

Moreover, for linear time-invariant (LTI) systems, (32) becomes

$$\mathcal{R}_{\boldsymbol{y},\eta}(\boldsymbol{\tau})_N = \left(\prod_{n=1}^{N} h^{(*)_n}(\tau_n) \right) \underset{\boldsymbol{\tau}}{\otimes} \mathcal{R}_{\boldsymbol{x},\xi}(\boldsymbol{\tau})_N, \quad (33)$$

where $h(\cdot)$ is the impulse-response function of the LTI system and

$$\beta_\eta(\boldsymbol{\tau}) = \alpha_\xi(\boldsymbol{\tau}). \quad (34)$$

Finally, as regards the systems performing a time-scale changing, the impulse-response function is

$$\mathfrak{h}(t, u) = \delta(u - at), \quad (35)$$

where $a \neq 0$ is the scale factor, the set Ω contains just one element, and

$$\psi_\sigma(f) = \frac{f}{a}, \qquad h_\sigma(t) = \frac{1}{|a|}\delta(t). \quad (36)$$

Therefore, from (25) it follows that

$$\beta_\eta(\boldsymbol{\tau}) = a\alpha_\xi(a\boldsymbol{\tau}), \qquad \xi \in W_x, \quad (37)$$

and hence (26) reduces to

$$\mathcal{R}_{\boldsymbol{y},\eta}(\boldsymbol{\tau})_N = \mathcal{R}_{\boldsymbol{x},\xi}(a\boldsymbol{\tau})_N. \quad (38)$$

REFERENCES

[1] W.A. Gardner and C.M. Spooner, "The cumulant theory of cyclostationary time-series, Part I: Foundation," *IEEE Trans. Signal Processing*, vol. 42, pp. 3387-3408, December 1994.

[2] C.M. Spooner and W.A. Gardner, "The cumulant theory of cyclostationary time-series, Part II: Development and applications," *IEEE Trans. Signal Processing*, vol. 42, pp. 3409-3429, December 1994.

[3] A. Napolitano, "Cyclic higher-order statistics: input/output relations for discrete- and continuous-time MIMO linear almost-periodically time-variant systems," *Signal Processing*, vol. 42, pp. 147-166, March 1995.

[4] L. Izzo and A. Napolitano, "Effects of nonrandom linear time-variant systems on higher-order cyclostationarity," in *Proc. of the Fifteenth GRETSI Symposium on Signal and Image Processing*, Juan-les-Pins, F, September 1995.

[5] L. Izzo and A. Napolitano, "Effects of random linear transformations on higher-order cyclostationary time-series," in *Proc. of Twenty-Ninth Annual Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, October 1995.

[6] H.L. Hurd, "Correlation theory of almost periodically correlated processes," *Journal of Multivariate Analysis*, vol. 37, pp. 24-45, April 1991.

[7] D. Dehay and H.L. Hurd, "Representation and estimation for periodically and almost periodically correlated random processes," in *Cyclostationarity in Communications and Signal Processing*, W.A. Gardner, Ed., pp. 295-326, New York, IEEE Press, 1994.