DETERMINING THE FALSE-ALARM PERFORMANCE OF HOS-BASED QUADRATIC PHASE COUPLING DETECTORS

J W A Fackrell * *and S McLaughlin* †
Dept of Electrical Engineering, University of Edinburgh, Edinburgh, EH9 3JL, UK.
Tel: +44 131 650 5655; fax: +44 131 650 6554
email: jwaf@ee.ed.ac.uk

ABSTRACT

Quadratic Phase Coupling (QPC) can be detected using Higher Order Statistics (HOS) measures. Previously, the bispectrum, biphase and bicoherence have been used as components in two QPC-detection algorithms. In this paper it is shown that the expressions which describe these detectors reduce to the same form for the white Gaussian noise case. The performance of these detectors is discussed, and particular attention is given to false alarms, which occur when QPC is detected in signals which do not exhibit QPC. A simple expression is derived which gives the probability of false alarm (P_{FA}) for QPC detectors. This expression shows how the P_{FA} increases as the Signal to Noise Ratio decreases, a relationship which is also observed in a simulation example.

1 INTRODUCTION

A signal generated by a nonlinear system contains information about that system, and using this information to identify the type of nonlinearity remains an interesting topic in signal processing research. HOS methods can be useful for this task, and in particular third-order HOS measures, such as the bispectrum and bicoherence, can be used to identify quadratic nonlinearities.

Under certain conditions [1] the *magnitude* of the bicoherence can be used to detect QPC [2]. However, if these conditions do not hold, then the *phase* of the bispectrum (the biphase) must also be used. Indeed two QPC detectors based on the biphase have recently been proposed [1] and [3,4]. These detectors have been put together in totally different ways, and appear, on the surface, to be quite different. The reconciliation of these two detection algorithms is a central theme of this paper. It is also shown that the probability of false alarm for these detectors can be easily derived. This is a new result which should be of practical use to those intending to implement biphase-based QPC tests.

Section 2 reviews QPC, and shows how a simple sinusoidal signal can be used to illustrate the QPC-detecting properties of the bispectrum and biphase. The structure of a statistical test for QPC is also described. Section 3 then describes the two recently proposed QPC detectors, and demonstrates that they reduce to the same form for signals composed of real sinusoids in additive white Gaussian noise (AWGN).

*JWAF is supported by BT and EPSRC.
†SML is supported by the Royal Society.

Section 4 then describes how the False-Alarm characteristics of these detectors can be found, and Section 5 compares the theoretical detector performance with simulation results in varying SNR conditions.

2 QUADRATIC PHASE COUPLING (QPC)

Quadratic Phase Coupling is a phenomenon associated with quadratic nonlinearities of the type shown in Fig. 1 below. If the input $s(n)$ to this system contains components with

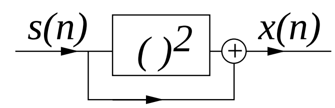

Figure 1: Simple system whose output exhibits QPC.

frequencies f_1 and f_2, it is easy to show that the output $x(n)$ will contain components with frequencies $f_1, 2f_1, f_2, 2f_2, f_1+f_2$ and f_1-f_2 as well as a DC component. Importantly, in $x(n)$ the phases at some frequencies will be related to the phases at other frequencies ; $\phi(2f_1) = 2\phi(f_1), \phi(2f_2) = 2\phi(f_2), \phi(f_1-f_2) = \phi(f_1)-\phi(f_2)$ and $\phi(f_1+f_2) = \phi(f_1)+\phi(f_2)$. This last relation can be detected by the bispectrum, defined as

$$B_{k,l} = X_k X_l X^*_{k+l}, \tag{1}$$

where $X_k = \frac{1}{M}\sum_{n=0}^{M-1} x(n)\exp(-j2\pi kn/M)$ is the $M-$point DFT of $x(n)$, and k and l are discrete frequencies. A simpler signal which also exhibits QPC is given by

$$x(n) = y(n)+v(n) = \sum_{i=1}^{3} A_i \cos(2\pi f_i n + \phi_i) + v(n), \tag{2}$$

in which $v(n)$ is additive noise with variance σ_v^2, and $\phi_i = U(-\pi,\pi](i=1,2)$. $x(n)$ exhibits frequency coupling if $f_3 = f_1+f_2$ and exhibits phase coupling if $\phi_3 = \phi_1+\phi_2$.

2.1 Detecting QPC using the Bicoherence

The squared bicoherence $b^2_{k,l}$ is a normalised bispectrum which has several useful properties (see [5] for a review), and is defined as

$$b^2_{k,l} = \frac{|E[X_k X_l X^*_{k+l}]|^2}{E[|X_k X_l|^2]E[|X_{k+l}|^2]}. \tag{3}$$

If multiple independent realisations of $x(n)$ are available, then the magnitude of the squared bicoherence estimate $\hat{b}^2_{k,l}$

can be used to quantify the extent of QPC at bifrequency (k, l) [2]. However, if only a single record is available, or if the multiple realisations are not independent, then the magnitude measure can be ambiguous unless the *phase* of the bispectrum $\Phi_{k,l}$ is utilised as well [1, 3, 4].

2.2 Detecting QPC using the Biphase

The biphase $\Phi_{k,l} = \angle B_{k,l} = \arctan \frac{Im[B_{k,l}]}{Re[B_{k,l}]}$ is the phase of the bispectrum. It is easy to show [2] that if there is QPC at (k, l) then $\Phi_{k,l} = 0$. Therefore at each (k, l) pair a test for QPC involves making a decision between the two alternative hypotheses;

H_0 : $\Phi_{k,l} = 0$: there is QPC.
H_1 : $\Phi_{k,l} \neq 0$: there is no QPC.

(For clarity the shortened notation $\Phi \equiv \Phi_{k,l}$, $b^2 \equiv b^2_{k,l}$ will be used in what follows, but it is important to note that the QPC test can be carried out at *each* bifrequency (k, l).)

For finite data lengths the biphase estimates $\hat{\Phi}$ will have non-zero variance, and so the test is recast as a test for *small* $\hat{\Phi}$ rather than *zero* Φ.

The test is formulated as follows;

1. Establish the statistical distribution of the estimated biphase $\hat{\Phi}$.
2. Choose a significance level α, which in turn defines a critical value Φ_c for the phase, such that $P[|\hat{\Phi}| > \Phi_c] = \alpha$.
3. Compare the estimated biphase with the critical value Φ_c and reject the null hypothesis if $|\hat{\Phi}| > \Phi_c$.

The test can be visualised in the complex $Re[B_{k,l}]$ v.$Im[B_{k,l}]$ plane as shown in Fig. 2. If the biphase lies within the shaded region then the null hypothesis H_0 is accepted.

The above procedure has been followed in two separate attempts to implement a biphase-based QPC detector. The approaches differed in the way in which step 1 was implemented; Fackrell et al [1] used empirical biphase distributions from periodogram-averaged bispectra estimates [6], which suggested that $\hat{\Phi}$ is approximately Normally distributed, while Zhou et al [3,4] used their own theoretical results which showed that the single-segment biphase estimate is approximately asymptotically Normal.

Both approaches determine the value of the critical phase Φ_c in Step 2 above from

$$\Phi_c = \sqrt{\Gamma(\alpha)\sigma^2_{\hat{\Phi}}} \tag{4}$$

where $\Gamma(\alpha)$: $P\left[\chi^2_1 > \Gamma(\alpha)\right] = \alpha$, and $\sigma^2_{\hat{\Phi}}$ is the variance of the biphase estimate. However, it is difficult to quantitatively compare the two approaches, since they use quite different expressions for $\sigma^2_{\hat{\Phi}}$; in [1] the biphase variance $\sigma^2_{\hat{\Phi}}$ is described in terms of the bicoherence magnitude b^2, but in [3,4] $\sigma^2_{\hat{\Phi}}$ is described in terms of the signal model parameters A_i, σ^2_v. The situation is further complicated by the fact that one detector [1] is based on a real signal model, while the other is based on a complex signal model [3, 4].

3 RECONCILING THE TWO DETECTORS

In this section it is demonstrated that, for the case of signals like $x(n)$ in Eqn. 2, with real sinusoids in AWGN (additive

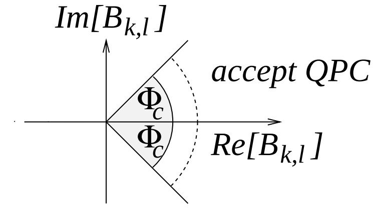

Figure 2: Schematic diagram showing complex bispectrum plane. Φ_c defines the critical region inside which H_0 is accepted.

second-order white Gaussian noise), the two detectors are in fact the same. This is shown by reformulating the expression for $\sigma^2_{\hat{\Phi}}$, the biphase variance at (f_1, f_2) from [1, 6], in terms of the model parameters A_i and the noise variance σ^2_v used in [3, 4].

For clarity the abbreviation FMW will be used to denote the Fackrell-McLaughlin-White detector [1] and ZGS will be used to denote the Zhou-Giannakis-Swami detector [3, 4].

The two expressions to be reconciled are;

1. The empirical variance of the biphase estimate at (f_1, f_2) computed for a real signal over K independent frames [6];
$$\sigma^2_{\hat{\Phi}}[\mathrm{FMW}] = \frac{1}{2K}\left(\frac{1}{b^2_{f_1,f_2}} - 1\right), \tag{5}$$
2. The corresponding theoretical variance of the biphase estimate at (f_1, f_2) for a single record (of length M) of a signal consisting of complex harmonics in noise. This is the asymptotic variance of $\sqrt{M}(\hat{\Phi} - \Phi)$ [3, 4] divided by M,
$$\sigma^2_{\hat{\Phi}}[\mathrm{ZGS}] = \frac{1}{2M}\sum_{i=1}^{3}\frac{\sigma^2_v}{A^2_i}. \tag{6}$$

Consider the real signal model of Eqn. 2. For the purpose of this illustration it is assumed that the additive noise $v(n)$ is AWGN with variance σ^2_v, but what follows should be equally applicable to any second-order white noise with a symmetric pdf.

3.1 Noise free case

If there is no noise $\sigma^2_v = 0$ then it is easy to show that $X_k = Y_k$, and that at bifrequency $(k = f_1, l = f_2)$ the expected values of the components of the bicoherence (Eqn. 3) will be [5];

$$\begin{aligned} |E[X_{f_1}X_{f_2}X^*_{f_3}]|^2 = |E[Y_{f_1}Y_{f_2}Y^*_{f_3}]|^2 &= \prod_{i=1}^{3}\frac{A^2_i}{4}, \\ E[|X_{f_1}X_{f_2}|^2] = E[|Y_{f_1}Y_{f_2}|^2] &= \prod_{i=1}^{2}\frac{A^2_i}{4}, \\ E[|X_{f_3}|^2] = E[|Y_{f_3}|^2] &= \frac{A^2_3}{4}, \end{aligned} \tag{7}$$

from which it easily follows that $b^2_{f_1,f_2} = 1$.

3.2 Noisy case

If there is some additive background noise $\sigma^2_v \neq 0$ then the DFT becomes $X_k = Y_k + V_k$. The numerator of the bicoherence is unchanged;

$$|E[X_{f_1}X_{f_2}X^*_{f_3}]|^2 = |E[Y_{f_1}Y_{f_2}Y^*_{f_3}]|^2 = \prod_{i=1}^{3}\frac{A^2_i}{4}, \tag{8}$$

since the cross terms between signal and noise which arise all have zero expectations, and the noise has zero expected bispectrum since it is assumed Gaussian. However, the two terms on the denominator do get affected by noise; the first term on the denominator becomes

$$\begin{aligned}
E[|X_{f_1}X_{f_2}|^2] &= E[|(Y_{f_1}+V_{f_1})(Y_{f_2}+V_{f_2})|^2] \\
&= E[|Y_{f_1}Y_{f_2}+Y_{f_1}V_{f_2}+V_{f_1}Y_{f_2}+V_{f_1}V_{f_2}|^2] \\
&= E[Y_{f_1}^2Y_{f_2}^2+Y_{f_1}^2V_{f_2}^2+Y_{f_2}^2V_{f_1}^2] \\
&= E[Y_{f_1}^2Y_{f_2}^2]+\frac{\sigma_v^2}{M}\left(E[Y_{f_1}^2]+E[Y_{f_2}^2]\right) \\
&= \prod_{i=1}^{2}\frac{A_i^2}{4}+\frac{\sigma_v^2}{M}\sum_{i=1}^{2}\frac{A_i^2}{4}, \qquad (9)
\end{aligned}$$

where the independence of the noise $v(n)$ from the sinusoids has been used, and the fact that, for second-order white noise $E[V^2(k)] = \sigma_v^2/M$. The second term on the denominator becomes

$$E[|X_{f_3}|^2] = E[|Y_{f_3}+V_{f_3}|^2] = E[Y_{f_3}^2]+\frac{\sigma_v^2}{M} = \frac{A_3^2}{4}+\frac{\sigma_v^2}{M}. \quad (10)$$

The theoretical bicoherence $b^2_{f_1,f_2}$ from Eqn. 3 is then given by combining Eqns. 8-10. This can then be substituted in Eqn. 5 to give a new expression for $\sigma^2_{\hat{\Phi}}[\mathrm{FMW}]$;

$$\sigma^2_{\hat{\Phi}}[\mathrm{FMW}] = \frac{1}{2KM}\left(\sum_{i=1}^{3}\frac{4\sigma_v^2}{A_i^2}+\frac{16\sigma_v^4}{M}\left[\frac{1}{A_2^2A_3^2}+\frac{1}{A_1^2A_3^2}\right]\right),$$

and provided M is reasonably large, the square bracketed term can be neglected to give

$$\sigma^2_{\hat{\Phi}}[\mathrm{FMW}] \approx \frac{1}{2KM}\sum_{i=1}^{3}\frac{4\sigma_v^2}{A_i^2} = \frac{4\sigma_v^2}{2KM}\sum_{i=1}^{3}\frac{1}{A_i^2}. \qquad (11)$$

Now from the asymptotic distribution of the biphase of a signal composed of *complex* harmonics $y(n) = \sum_i A_i \exp(-j2\pi f_i n+\phi_i)$ in *complex* noise [3, 4], an expression for *real* signals such as Eqn 2 can be formed. The results derived in [3, 4] are (we believe) still valid, as long as each instance of the squared sinusoid amplitude A_i^2 is scaled down by a factor of 4. This is because $E[|X_{f_i}|^2] = A_i^2$ for a complex harmonic signal [3, Eqn.24], but $E[|X_{f_i}|^2] = A_i^2/4$ for a real signal (the total energy of the real signal is half that of the complex signal, and the real signal has half of its energy mirrored above the folding frequency). The biphase variance is then given by

$$\sigma^2_{\hat{\Phi}}[\mathrm{ZGS}] \Rightarrow \frac{1}{2M}\sum_{i=1}^{3}\frac{\sigma_v^2}{A_i^2/4} = \frac{4\sigma_v^2}{2M}\sum_{i=1}^{3}\frac{1}{A_i^2}. \qquad (12)$$

Furthermore, if K independent realisations are available, then the biphase variance is scaled down by a factor of K. The asymptotic biphase expression then leads to the following expression for the variance of the biphase averaged over K independent segments of $x(n)$;

$$\sigma^2_{\hat{\Phi}}[\mathrm{ZGS}] \Rightarrow \frac{4\sigma_v^2}{2KM}\sum_{i=1}^{3}\frac{1}{A_i^2}. \qquad (13)$$

The equivalence of Eqns. 11 and 13 shows that for the AWGN case, the two QPC detectors are the same. In other words the FMW detector is a special case of the more general ZGS detector.

4 ERRORS IN QPC DETECTORS

Having shown that the two QPC detectors are fundamentally the same, the matter of false alarm performance of the detectors will now be considered.

In any hypothesis testing situation, there are two types of error which can occur [7];

> **Type I error:** The hypothesis is true but is rejected. The probability of a Type I error is measured by α, the significance level of the test.
>
> **Type II error:** The hypothesis is false but is accepted. The probability of this type of error is measured by β, the *power* of the test.

For QPC detection, Type I errors occur when there is QPC, but H_0 is rejected (i.e. $P\left[|\hat{\Phi}| > \Phi_c \middle| \mathrm{QPC}\right] = \alpha$), and Type II errors occur when there is no QPC, but H_0 is accepted (i.e. $P\left[|\hat{\Phi}| < \Phi_c \middle| \text{no QPC}\right] = \beta$).

Now the two previous applications of phase-based QPC detectors have used a chosen α level (e.g. $\alpha = 0.05$) to determine the critical biphase value, but only [1] very briefly discusses the *power* of the test. Zhou et al [3, 4] use the term "Probability of False Alarm P_{FA}" to describe α. We believe a more useful definition of P_{FA} is "the probablity of detecting QPC when there is in fact no QPC present", i.e. $P_{FA} \equiv \beta$.

In practical applications the probabilities of both these types of errors must be evaluated, since they present conflicting requirements. Indeed discussion of the detector without reference to the probability of Type II errors is meaningless, because a detector which detects QPC **all the time** (regardless of whether or not the signal exhibits QPC) will have the apparently "ideal" performance of $P[\text{Type I}] = 0$. The failings of such a detector would only manifest themselves in $P[\text{Type II}]$.

4.1 P_{FA} for QPC detectors

Now it turns out that β, the probability of a Type II error, can be determined easily. As discussed in Section 2, the critical phase value Φ_c defines a QPC-acceptance region in the complex $Re[B(k,l)]$ v.$Im[B(k,l)]$ plane, as shown in Fig. 2. The width of the acceptance region is $2\Phi_c$ in which the factor of 2 accounts for both negative and positive angles.

The probability of a Type II Error is the probability that, if there is no QPC, $\hat{\Phi}$ falls within this acceptance region. Now if there is no QPC, Φ, and thus its estimate $\hat{\Phi}$, will be uniformly distributed as $U(-\pi,\pi]$, so β is given by

$$\begin{aligned}
\beta &= \frac{\text{angular width of } H_0 \text{ acceptance region}}{2\pi} = \frac{\Phi_c}{\pi} \\
&= \frac{1}{\pi}\sqrt{\Gamma(\alpha)\sigma^2_{\hat{\Phi}}}. \qquad (14)
\end{aligned}$$

where Φ_c is given by Eqn. 4 and $\sigma^2_{\hat{\Phi}}$ is given by either Eqn. 5 or Eqn. 6. Thus the probabilities of Type I and Type II errors (α and β respectively) are linked through Eqn 14.

It can be seen, that for a given α, β will vary according to $\sigma^2_{\hat{\Phi}}$. Now $\sigma^2_{\hat{\Phi}}$ is closely related to the signal to noise (SNR) ratio, so we see that as the SNR varies, so the performance of the detector will vary. This is illustrated in the simulations which follow.

It is worth noting here, that since Φ_c can exceed π, so β can exceed unity. This situtation, which is meaningless from a probabilistic viewpoint, will occur at low SNR's. In other words, if the SNR is small, then Φ_c will be so large that *all* estimated phases will fall within the acceptance region, and although P[Type I] = 0 in such a case, P[Type II]=1 also, so the detector will be rendered useless.

5 SIMULATIONS

To see how well Eqn. 14 predicts the P_{FA} for a real signal, a simulation experiment will be described. The simulation described here is a real-signal version of that presented by Zhou et al [3]. The signal model is Eqn. 2 with $M = 1024$, $f_1 = 0.4$, $f_2 = 0.9$, $f_3 = f_1 + f_2 = 1.3$, $f_s = 2\pi$, $A_i = 1 (i = 1, 2, 3)$, $\phi_i = U(-\pi, \pi] (i = 1, 2)$. The two signal types considered are;

case 1 : There is QPC $\phi_3 = \phi_1 + \phi_2$,
case 2 : There is **no** QPC $\phi_3 = U(-\pi, \pi]$.

The performance of the detector for these two cases is measured by counting the number (N_{QPC} and $N_{\mathrm{no\ QPC}}$ respectively), of detections (or "hits") at the bifrequency (f_1, f_2) in 200 Monte-Carlo runs for different SNR's (SNR = $10\log_{10}(\sigma_y^2/\sigma_v^2)$ where σ_y^2, σ_v^2 are the variances of $y(n)$ and $v(n)$ respectively), where the noise is AWGN.

For this example $\alpha = 0.05$, $\Gamma(\alpha) = 3.841$, and the theoretical P_{FA} β is given by Eqns. 12 and 14 with $K = 1$ as

$$\begin{aligned} \beta &= \frac{1}{\pi}\sqrt{\frac{2\Gamma(\alpha)\sigma_v^2}{M}\sum_{i=1,2,3}\frac{1}{A_i^2}} \\ &= \frac{3}{\pi}\sqrt{\frac{\Gamma(\alpha)}{M 10^{\mathrm{SNR}/10}}}, \end{aligned} \tag{15}$$

since the variance of a real sine wave of amplitude A is $A^2/2$. As this is a single-record detector ($K = 1$) the ZGS formulation is used in which A_i and σ_v^2 in Eqn. 12 are replaced by their estimates $\hat{A}_i$ and $\hat{\sigma}_v^2$ (as in [3,4]). For $K = 1$ the FMW formulation does not work, since $\hat{b}^2$ the estimate of b^2 in Eqn. 5 is identically unity if $K = 1$.

Fig. 3 shows the performance of the single-record detector in both the cases described above (i.e. with QPC and without QPC). The plot shows the following;

$$\begin{aligned} \text{Theory}: \quad & P\,[\mathrm{hit|QPC}] && = 1 - \alpha = 0.95, \\ \text{Experiment}: \quad & P\,[\mathrm{hit|QPC}] && = \frac{N_{\mathrm{QPC}}}{200}, \\ \text{Theory}: \quad & P\,[\mathrm{hit|noQPC}] && = \beta(\text{Eqn. } 15), \\ \text{Experiment}: \quad & P\,[\mathrm{hit|noQPC}] && = \frac{N_{\mathrm{no\ QPC}}}{200}. \end{aligned} \tag{16}$$

The results show that at high SNR's the detector works well, with P[hit|QPC] $\approx$ 0.95, and P[hit|no QPC] $\approx$ 0. As the SNR is reduced two things happen; P[hit|QPC] falls, and, importantly, P[hit|no QPC] rises. Equation 15 predicts the rise in P[hit|no QPC] well, but the reason for the fall of P[hit|QPC] is not clear. It is thought that this might be due to some loss of validity in the approximations which are used to form the test. Alternatively it could be a reflection of the fact that as the SNR falls the estimates of the sinusoid amplitudes needed in Eqn. 13 will become less accurate. This remains a matter for further research.

It is thus evident that the single-record phase detector is not "fail-safe" - i.e. at low SNR it tends to detect QPC erroneously in signals which do not in fact exhibit QPC.

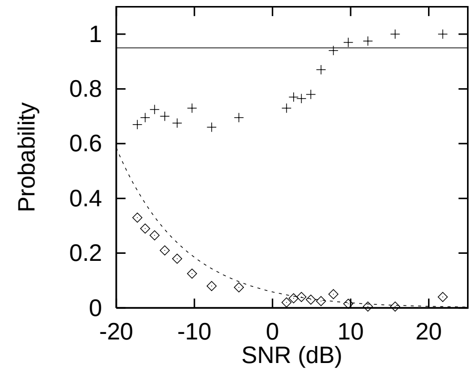

Figure 3: Comparison between theory and experiment for the single-record QPC detector. P[hit|QPC]; theory (solid line) and experiment (crosses). P[hit|no QPC]; theory (dashed line) and experiment (diamonds).

6 CONCLUSIONS

It has been shown that two recently proposed QPC detectors are, to a first approximation, identical for the case of real sinusoids in AWGN. It has been shown that the Probability of Type II errors occurring in the detector can be easily found from the expressions which describe the detector, and that the Probability of Type II errors has an inverse relationship with the SNR. Further work is warranted to find out if the detector can be improved if different techniques are used to estimate the sinusoid amplitudes and noise variance.

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