HIGHER ORDER DETECTION TEST FOR DETERMINISTIC SIGNALS

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ABSTRACT

In the contex of electromagnetic signals, we want to detect a transient in a non stationnary gaussian noise by a higher order statistic test. In this paper, we use a new formalism (an extension of Gardner's work) that enables us to evaluate theoretically the response of higher order statistic test for detection. We develop the theoretical ground and we prove that higher order statistic detection test provides a very short delay detection. We apply our methods to simulation of a simple and typical example : the kurtosis.

INTRODUCTION

In the study of electromagnetic signals, our aim is to detect a deterministic transient signal $s(t)$, in presence of an additive gaussian noise $n(t)$. The noise has a slowly moving spectrum and we can consider the noise variance as a constant during all the observation. Second order methods are difficult to use as they need a prewhitening. We propose to use the Higher Order Statistic (HOS). The HOS have already been used for detection [1]. Especially the normalized cumulant of order 3 and 4 [2]. The HOS allows us to change of observation space in order to make the detection easier.
The choice between the two hypothesis :

$$\left\{ \begin{array}{lcl} H_0 & : & x(t) = n(t) \\ H_1 & : & x(t) = s(t) + n(t) \end{array} \right.$$

where $n(t)$ is a gaussian noise and $s(t)$ is the signal to detect.
is then transposed into the new formulation :

$$\left\{ \begin{array}{lcl} H_0 & : & \zeta_n = 0 \\ H_1 & : & \zeta_n \neq 0 \end{array} \right.$$

where ζ_n is normalized cumulant of order n.
We propose here a sequential detection. First, we extend the work of Gardner for deterministic signals to noisy deterministic signals, in order to have a theoretical knowledge of the shape of the kurtosis response. Then we are be able to evaluate the HOS test in terms of time delay detection of and of time location of a transient.

1 HIGHER ORDER FOR DETERMINISTIC SIGNALS

This part presents an extension of the non probabilistic theory of Gardner [3] [4]. In straight analogy with conventional probability distribution function (PDF), Gardner defines a Fraction of Time (FOT) PDF of the real time serie $x(t)$. We present a interpretation of the FOT PDF for a deterministic signal, which enables us to compute the cumulant of a noisy deterministic signal.

1.1 Analogy with ergodic signals

For ergodic signal the expected values (ensemble averages) are time invariant and are equal to time averages.

$$E\{x\} = \int_{-\infty}^{\infty} u \, p_x(u) du = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t) dt$$

For ergodic signals the probability distribution for the amplitude of a time serie is then :

$$\begin{array}{rcl} F_x(u) & \hat{=} & \text{probability that } x(t) < u \\ & = & \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} U[u - x(t)] dt \end{array} \quad [1]$$

The PDF of the signal is then :

$$p_x(u) = \frac{dF_x(u)}{du}$$

The probability law can be then envisionned as being derived from relative frequencies of occurence of events so that probability functions are really *fraction of time distributions.*

1.2 Definition of FOT PDF for deterministic signals

It is then easy to transpose these time average tools to deterministic signals. For a deterministic signal, $s(t)$, the FOT PDF can be defined as :

$$p_s(u) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \delta(u - s(t)) dt \qquad (1)$$

[1] U is the unit-step function :

$$U(u) = \left\{ \begin{array}{ll} 1 & u > 0 \\ 0 & u \leq 0 \end{array} \right.$$

We can then define tools such as first and second characteristic functions, and higher order moments and cumulants for deterministic signal. Moments of a deterministic signal are then defined as :

$$\mu_n\{s\} = \int_{\infty}^{\infty} u^n \, p_s(u) du = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} s(t)^n \, dt \quad (2)$$

The relation between cumulants and moments are the same as those for random signals. We have for zero mean signals the following relations for orders 1 to 4 :

$$\zeta_2\{s\} = \mu_2\{s\} \quad (3)$$
$$\zeta_3\{s\} = \mu_3\{s\} \quad (4)$$
$$\zeta_4\{s\} = \mu_4\{s\} - 3\mu_2\{s\}^2 \quad (5)$$

For non periodic signals the transposition is not accurate. For example if we take the signal of finite ernergy $s(t) = \frac{1}{1+t^2}$ its FOT PDF is $p_s(u) = \delta_0(u)$. We have then a great loss of information. In this paper we will stick to periodic signals and signals of finite duration which can be periodised.

1.3 Probabilistic interpretation of the FOT PDF

If we study the random variable $S = s(\theta)$, where θ is a uniform random variable between $[-\frac{T}{2}, \frac{T}{2}]$ (where T is the period of the signal $s(t)$). If we define the ensemble $D_u = \{t | s(t) < u\}$. As θ is uniform random variable, we have :

$$\begin{aligned} Prob(X < u) &= Prob(\theta \in D_u) \\ &= \text{measure of } D_u/T \end{aligned}$$

Which if we look at figure 1, we see that :

$$Prob(X \leq u) = \begin{cases} 0 & u < A \\ \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} U(u - s(t))dt & A \leq x \leq B \\ 1 & u > B \end{cases}$$

where $s : [-\frac{T}{2}, \frac{T}{2}] \to [A, B]$. As we have a periodic signal $\lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} U(u - s(t))dt = \int_{\frac{-T}{2}}^{\frac{T}{2}} U(u - s(t))dt$. We have shown that the FOT PDF is the PDF of the

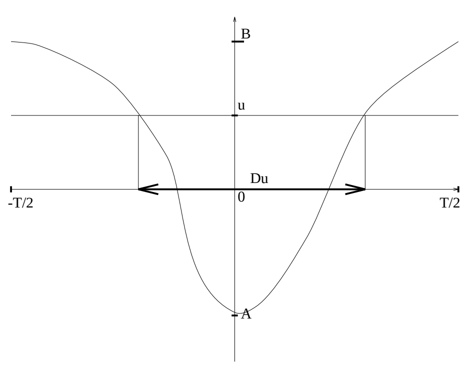

Figure 1: *Probabilistic interpretation of the FOT PDF*

random variable S. This means that the FOT PDF describes the random signal of unknown phase which can also be expressed as : the FOT PDF describes the random variable of the observed value when the time of observation is chosen at random. This is always the case in the context of surveillance, we don't know when the phenomenon we are looking for is likely to occur.

2 HOS FOR NOISY DETERMINISTIC SIGNALS

2.1 FOT PDF for noisy deterministic signals

In this part we consider the noisy signal : $x(t) = s(t) + n(t)$. Where $n(t)$ is a stationnary random signal of PDF p_n. We want to evaluate the FOT PDF of $x(t)$, so we need to study the random variable $X = S + N$ where $S = s(\theta)$ and $N = n(\theta)$ where θ is still a uniform random variable between $[-\frac{T}{2}, \frac{T}{2}]$. We can suppose that the random variables S and N are independant. This hypothesis is reasonnable when the physical sources of the noise are different from the signal physical sources. In this case X is the sum of two independant random variables, which means :

$$p_X = p_S * p_N$$

As $n(t)$ is a stationnary noise we have $p_n = p_N$. So the FOT PDF of $x(t)$ is :

$$p_x = p_s * p_n \quad (6)$$

We illustrate this by the FOT PDF of a sinusoid with uniform distributed noise. The FOT PDF of $s(t) = Asin(2\pi t)$ is

$$p_s(u) = \begin{cases} 0 & u < -A \; A < u \\ \frac{1}{\sqrt{1-u^2}} = \frac{d \, arcsin(u)}{du} & -A \leq u \leq A \end{cases}$$

(the expression of p_s is given in [5]). As it is shown in figure 2, the result obtained by the expression (6) is very similar to the result obtained by the histogram of a digital noisy sinusoid.

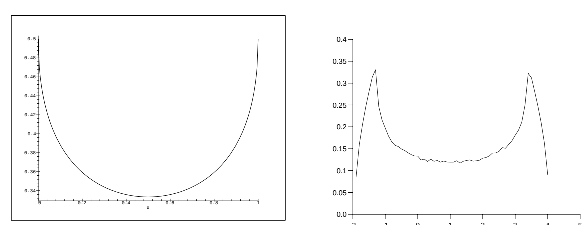

Figure 2: *Theoretical and estimated FOT PDF of a noisy sinusoid*

2.2 Cumulants of a noisy deterministic signal

As we have $p_x = p_s * p_n$, the signal $s(t)$ and the noise $n(t)$ behave as two independant random signals. So the cumulant of x is equal to cumulant of s plus the cumulant of n. This result generalize to noisy deterministic signals the well known result for random signals : HOS are insensitive to additive gaussian noise.

3 ANALYSIS OF THE KURTOSIS RESPONSE

We now consider only the cumulant of order 4. The same approach can be done for other orders, but the kurtosis seems to give better performances. In this part we will analyse theoretical response of the kurtosis and compare it to simulation result.

3.1 Computation of the kurtosis

We now calculate the kurtosis of a noisy transient signal in a sliding window. Which means at time t, we only study the signal between [t-T, t]. We can then consider we have a signal of finite duration. The previous results are still valid.

So the nth order moment of the signal $x(t)$ for T wide causal window is :

$$\mu_t^n\{x\} = \frac{1}{T}\int_{t-T}^{t} x(u)^n du$$

The cumulant of order 4 is cumputed by using the relation (5). To fix the detection threshold we need to normalized the cumulant. In order to have a better contrast between the noise and the signal out of the sensor, we prefer to normalize the fourth order cumulant by the noise variance, instead of the signal plus noise variance. So we suppose we have time to learn the noise variance σ_n^2 before the signal appears. The window length is small enough so we consider that the noise in the window is stationnary. Our sensor is then:

$$k(t) = \frac{Cum_t^4 x}{\sigma_n^4}$$

3.2 Theoretical and simulation results

We show here two examples of transients :

- a dipolar signal $e_1(t) = \frac{1-7t^2}{(1+t^2)^{5/2}}$.
- a triangle : $e_2(t) = \begin{cases} 0 & t < -\frac{1}{2}\, t > \frac{1}{2} \\ t+1/2 & -\frac{1}{2} \leq t < 0 \\ -t+1/2 & 0 \leq t < \frac{1}{2} \end{cases}$

For both these signals we have evaluated the optimal window length. Unlike second order statistics, the maximum response is not given when the window is centered on the transient. In order to evaluate the optimal window length, we need to plot the 3D function $k(t, T)$ and find the value T that maximizes this function .

3.2.1 signal : e_1

For the signal e_1, the optimal length is estimated at 1.6s. This is a very short time compare to the duration of the transient (fig. 3). If we look at the theoretical response of the kurtosis test (plain curve fig. 4) of this transient, we see that the kurtosis reaches its maximum response very soon after the signal appears in the analyzing window. This means that we detect very quickly a rupture in the process, this rupture is likely to have a very short duration as the optimal window is of 1.6s length. The kurtosis test has a very sharp response compared to the square energy response (dot curve in fig. 4). This enables us to do a better time location of the signal.

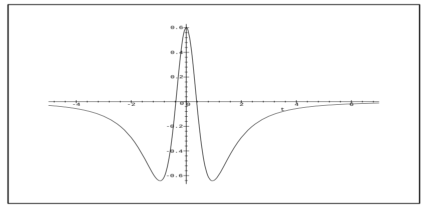

Figure 3: *signal* e_1

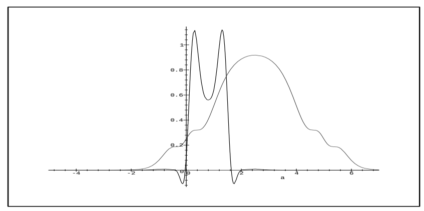

Figure 4: *Theoretical kurtosis response (with a* 1.6*s wide causal window) and square energy response*

We examine here an example with a simulated noise. Figure 5 show the signal $x(t)$ and that the transient starts at sample 750 and stops at sample 1250. The simulation results (fig. 6) are similar to what we expected from the theoretical ones. We still have a very fast detection and a very sharp kurtosis response.

3.2.2 signal : e_2

For the signal e_2, the optimum window length is $T = 0.5s$. The kurtosis reaches its maximum response (fig. 7) very soon after the signal appears in the analyzing window. Its response is symetrical, there is a second

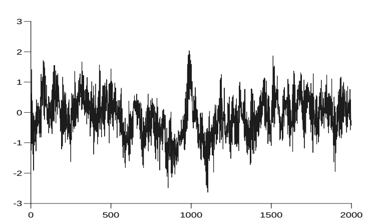

Figure 5: *signal* e_1

maximum response when the signal exits the analyzing window.

The transient of the noisy signal (fig. 8) starts at sample 1000 and stops at sample 2000. The simulation results (fig. 9) are similar to what we expected from the theoretical ones.

CONCLUSION

In this paper it is shown how to evaluate the theoretical response of the kurtosis of a transient signal. The analysis of this response, proves that the kurtosis is a very fast sensor and can give a very good time location of the transient. The simulation results agree with the theory. If we compare the results we have for different transients, the time delay depends on the shape of the transient we study. So in order to perform an efficient time location of a specific transient, we need to study the kurtosis response of this transient. These results have been extended to other HOS test such as skewness and neguentropy (which is defined in [1]) but kurtosis remains the best compromise beetween detection performances and computation time.

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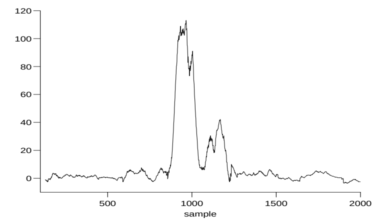

Figure 6: *Simulation kurtosis response (with a 130 sample wide causal window)*

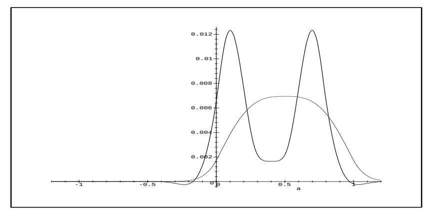

Figure 7: *Theoretical kurtosis response (with a .5s wide causal window) and square energy response*

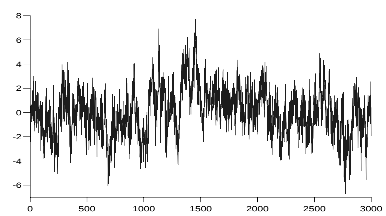

Figure 8: *signal* e_2

Figure 9: *Simulation kurtosis response (with a 500 sample wide causal window)*