DETECTION AND CLASSIFICATION OF NOISY AR AND ARMA PROCESSES

Jean-Yves TOURNERET, Karine VAREILLE and Martial COULON
ENSEEIHT/GAPSE, National Polytechnics Institute of Toulouse
2 rue Camichel, 31071 Toulouse, France
tel: (33) 61 58 83 67 Fax:(33) 61 58 82 37 email: tournere@len7.enseeiht.fr

ABSTRACT

The paper focuses on the detection and the classification of noisy AR and ARMA processes. These two kinds of processes cannot be distinguished by means of their second-order statistics, since they are Spectrally Equivalent (SE). Higher-order statistics are shown to be an efficient tool for their detection. A Neyman-Pearson (NP) test, based on these higher-order statistics, is then studied. The performance of the NP test provides a reference for comparing suboptimal detector performances.

1 INTRODUCTION

Parametric AR, MA or ARMA models have been used successfully in many signal processing applications. These models represent a given signal as the output of a linear filter driven by a Gaussian or non-Gaussian input. One interesting question is deciding whether a given signal is Gaussian or non-Gaussian, or the output of a linear or non-linear system. This question has been studied with increasing interest in recent years. However, a difficult remaining problem is to choose the model which best fits a given signal. This problem has been studied recently for the detection and the classification of deterministic signals in multiplicative and additive noise [6]. This paper studies the detection of Spectrally Equivalent (SE) noisy AR and ARMA processes. It is well known that additive noise corrupted AR models and ARMA models are SE [4]. Thus, neither the correlation function nor the power spectral density are well suited for the detection of noisy AR and ARMA processes. The first part of the paper shows that SE noisy AR and ARMA processes have different higher-order spectra. This property provides a tool for detecting and classifying these two processes. The second part of the paper develops the Neyman-Pearson (NP) detector for appropriate higher order cumulants of the two processes. The performance of the NP detector provides a reference for comparing suboptimal detector performances.

2 HIGHER-ORDER SPECTRA

Consider the two signal models:

- An AR(p) process $x(n)$ driven by a non-Gaussian input $e(n)$ and corrupted by a additive random noise $b(n)$:

$$y_1(n) = x(n) + b(n) \tag{1}$$

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + e(n) \tag{2}$$

  Noises $e(n)$ and $b(n)$ are assumed to be i.i.d. and mutually independent.

- An ARMA(p,p) process $y_2(n)$ driven by a random i.i.d. input $g(n)$ with the same mean and power spectral density as $y_1(n)$ i.e. :

$$y_2(n) = -\sum_{k=1}^{p} a_k y_2(n-k) + \sum_{k=0}^{p} b_k g(n-k) \tag{3}$$

  with:

$$\gamma_{2e} + \gamma_{2b}\,|A(z)|^2 = \gamma_{2g}\,|B(z)|^2 \tag{4}$$

  $A(z)$ and $B(z)$ denote the Z-Transform of the corresponding AR and MA parameters, γ_{ke},γ_{kb} and γ_{kg} are the kth-order cumulants of the processes $e(n)$, $b(n)$ and $g(n)$.

The kth-order spectrum of $y_1(n)$ and $y_2(n)$ are given by the two well-known expressions [5]:

$$S_{ky_1}(z_1,\dots,z_{k-1}) = \frac{\gamma_{ke} + \gamma_{kb} A(z_1)\dots A(z_{k-1})A((z_1\dots z_{k-1})^{-1})}{A(z_1)\dots A(z_{k-1})A((z_1\dots z_{k-1})^{-1})} \tag{5}$$

$$S_{ky_2}(z_1,\dots,z_{k-1}) = \gamma_{kg}\frac{B(z_1)\dots B(z_{k-1})B((z_1\dots z_{k-1})^{-1})}{A(z_1)\dots A(z_{k-1})A((z_1\dots z_{k-1})^{-1})} \tag{6}$$

Two cases are considered:

- *If $b(n)$ or $e(n)$, or both, are non-Gaussian*, there exists an order $k \geq 3$ such that $\gamma_{kb} \neq 0$ or $\gamma_{ke} \neq 0$. Since $y_1(n)$ and $y_2(n)$ have the same power spectrum, Eq. (5) and (6) show they cannot have the same kth-order spectra (see Appendix A for order 3).

- *If both $b(n)$ and $e(n)$ are Gaussian*, the noisy AR process is Gaussian. All higher-order spectra of $y_1(n)$ and of any Gaussian ARMA process $y_2(n)$ are zero. Thus, higher-order spectra cannot be used for the detection and the classification of SE noisy AR and ARMA processes.

3 NEYMAN-PEARSON TEST

Consider the detection of SE noisy AR and ARMA processes using the Neyman-Pearson test for the kth-order cumulants. Denote $C_{ky}(\tau)$ as a kth-order cumulant slice. Denote $\widehat{C}_{ky}(\tau)$ as the kth-order sample cumulant slice obtained by replacing expectations in $C_{ky}(\tau)$ by sample averages. According to section 2, $C_{ky}(\tau)$ is chosen differently for the noisy AR process and for its SE ARMA process. The two hypotheses can be expressed as follows:

$$\begin{array}{lll} H_0: & \text{(Noisy AR process)} & \widehat{C}_{ky}(\tau) \sim N(M_0, \Lambda_0) \\ H_1: & \text{(ARMA process)} & \widehat{C}_{ky}(\tau) \sim N(M_1, \Lambda_1) \end{array}$$

The kth-order sample cumulants of noisy AR or ARMA processes are asymptotically normally distributed [1]. The mean vectors and the covariance matrices are known functions of the model parameters [2]. The Neyman-Pearson test for deciding between H_0 and H_1 is given by:

H_0 rejected if

$$\sqrt{\frac{|\Lambda_0|}{|\Lambda_1|}} \exp -\frac{1}{2}\left(Q_1 - Q_0\right) > k_1 \tag{7}$$

Q_0, Q_1 in (7) are the quadratic forms:

$$Q_0 = (\widehat{C}_{ky}(\tau) - M_0)^T \Lambda_0^{-1} (\widehat{C}_{ky}(\tau) - M_0) \tag{8}$$

$$Q_1 = (\widehat{C}_{ky}(\tau) - M_1)^T \Lambda_1^{-1} (\widehat{C}_{ky}(\tau) - M_1) \tag{9}$$

Eq. (7) leads to:

$$H_0 \text{ rejected if } H = Q_0 - Q_1 > k_2 \tag{10}$$

Since $\Lambda_0 \neq \Lambda_1$ in the general case, the optimal Neyman Pearson detector is non-linear with respect to $\widehat{C}_{ky}(\tau)$. Consider the Cholesky factorizations of the symmetric-definite matrices Λ_0 and Λ_1:

$$\Lambda_0 = B_0.B_0^t \text{ and } \Lambda_1 = B_1.B_1^t$$

If the unit normal n-dimensional vector $V = [v_1, ..., v_n]^t$ is defined by:

$$V = B_0^{-1}(\widehat{C}_{ky}(\tau) - M_0) \quad \text{under } H_0 \tag{11}$$

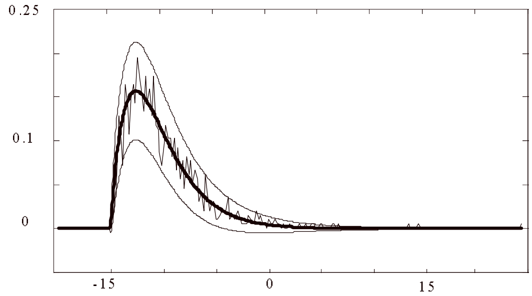
$$V = B_1^{-1}(\widehat{C}_{ky}(\tau) - M_1) \quad \text{under } H_1, \tag{12}$$

the test statistics H can be expressed as follows:

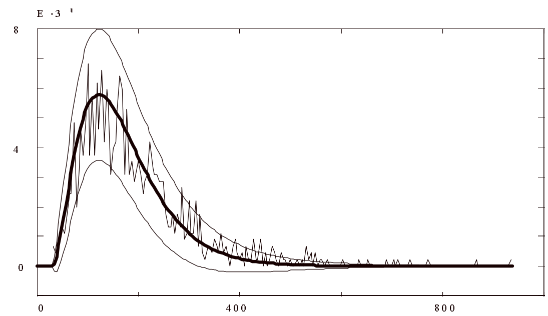
$$H = \sum_{k=1}^{n} \lambda_{0k} \left(v_k - a_{0k}\right)^2 \text{ under } H_0 \tag{13}$$

$$H = \sum_{k=1}^{n} \lambda_{1k} \left(v_k - a_{1k}\right)^2 \text{ under } H_1. \tag{14}$$

λ_{0k} (resp. λ_{1k}) denote the eigenvalues of matrices $E_0 = I_n - B_0^t \Lambda_1^{-1} B_0$ (resp. $E_1 = -I_n + B_1^t \Lambda_0^{-1} B_1$) and I_n is the identity matrix. Eq. (13) and (14) show that H is a weighted sum of shifted, squared and normalized one-dimensional Gaussian variables. It is a difficult problem to determine the theoretical statistics of H. Series expansions of mixtures of central and non central χ^2 distributions are available under specific conditions [3]. However, these expansions are difficult to study. Instead, approximations are used which lead to a simple test performance computation. Fig. 1.a and 1.b show a particular case where the probability density function (p.d.f.) of H can be approximated by a gamma distribution under hypotheses H_0 and H_1. Simulations have confirmed this approach in practical applications.



(a)



(b)

Fig. 1. Histograms and p.d.f. of H with 95% confidence interval a) under H_0 b) under H_1

In a one dimensional test ($n = 1$), note the distribution of H/λ_{01} is a non-central χ^2 distribution with 1 degree of freedom and non-centrality parameter a_{01}. Thus, the exact distribution of H is known. The exact ($n = 1$) or approximate ($n > 1$) statistics of H can be used to compute the performance of the optimal Neyman-Pearson test. The False-Alarm (PFA) and Non-Detection (PND) probabilities are functions of the threshold k_2. Let $f_0(t)$ and $f_1(t)$ denote the p.d.f. of H under hypothesis H_0 and H_1, respectively. Thus,

$$PND = \int_{k_2}^{+\infty} f_0(t)\, dt \tag{15}$$

$$PFA = \int_{-\infty}^{k_2} f_1(t)\,dt \tag{16}$$

4 SIMULATION RESULTS

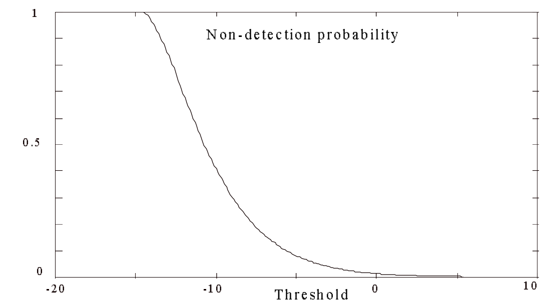
For simplicity, consider an AR(1) model with parameters [1; −0.5] driven by a zero mean iid exponentially distributed input $e(n)$ with $\sigma_e^2 = 1$ and $\gamma_{3e} = 2$. The corresponding AR(1) process is corrupted by a zero mean Gaussian sequence with a signal-to-noise ratio:

$$SNR = 10\log_{10}\left(\frac{P_x}{P_b}\right) = -0.5dB \tag{17}$$

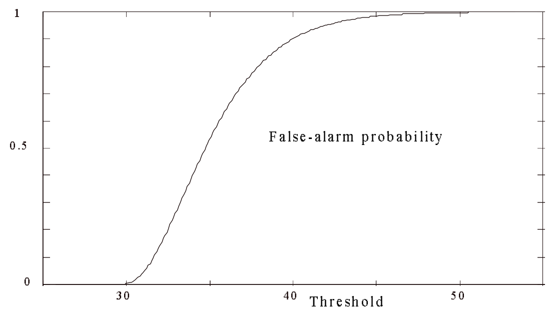
P_x and P_b are the power of the signal $x(n)$ and noise $b(n)$, respectively. Appendix B presents theoretical expressions for the third order cumulants of the noisy AR and the SE ARMA processes (note both processes are zero mean). The cumulant lag $\tau = (\tau_1, \tau_2)$ maximizing $D(\tau) = [C_{3y_1}(\tau_1,\tau_2) - C_{3y_2}(\tau_1,\tau_2)]^2$ is well-suited for the detection of the two processes. The lag $\tau = (0,0)$ corresponds to the lag for the maximum of $D(\tau)$. The following slice (corresponding to $\tau_1 = 0$) has then been chosen:

$$C_{3y} = [C_{3y}(0,0), ..., C_{3y}(0,3)]^t \tag{18}$$

Fig. 2 displays the Neyman-Pearson test non-detection and false-alarm probabilities versus the threshold k_2 for $N = 1024$ data samples. The corresponding ROC curves are shown in Fig. 3 for different SNR.



(a)



(b)

Fig. 2 : Performance of the Neyman-Pearson test as a function of the threshold k_2 a) PND *b) PFA*

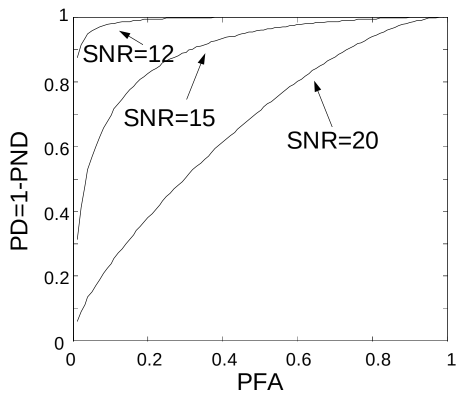


Fig. 3 : ROC Curves versus SNR

Figs. 2, obtained for $SNR = 10dB$, indicate that the detector performs well. The ROC curves, displayed in Fig. 3, show that the Neyman-Pearson test improves when the SNR decreases. This is different from the usual case. When the SNR is small, the Gaussian noise power is large. The distribution of the noisy AR process is then very close to the Gaussian. Thus, it is easy to distinguish the noisy AR process from its SE non-Gaussian ARMA process. On the other hand, when the SNR is large, the Gaussian noise power is small. The noisy AR process tends to be statistically equivalent to its SE ARMA process. Thus, it is hard to distinguish the two processes.

The Neyman-Pearson detector is optimal in the sense that it minimizes the non-detection probability for a fixed false-alarm probability. The detector provides a performance reference for comparing suboptimal detectors. The noisy AR and ARMA parameters are estimated in practical applications by anyone of a number of second or higher-order methods [5].

5 CONCLUSION

In conclusion,

- Higher order spectra can distinguish between SE additive noise corrupted AR models and ARMA models.
- Neyman-Pearson test simulation results show the efficiency of this method of distinguishing between the two processes.

6 Appendix A : Higher Order Spectra of SE Noisy AR and ARMA Processes

This appendix shows SE noisy AR and ARMA processes cannot have the same higher order spectra if $b(n)$ or $e(n)$, or both, are not Gaussian. For simplicity, the study is restricted to noisy AR processes with $\gamma_{3b} \neq 0$. However, the study could be extended to other processes, as long as $b(n)$ and $e(n)$ are not simultaneously

Gaussian. SE noisy AR and ARMA processes are shown to have different third-order spectrum, for $\gamma_{3b} \neq 0$. This implies that Eqs. (19) and (20) below cannot be satisfied simultaneously (note that Eqs. (19) and (20) emphasize that $y_1(n)$ and $y_2(n)$ have the same power spectral density and the same third order spectrum)

$$\gamma_{2e} + \gamma_{2b} |A(z)|^2 = \gamma_{2g} |B(z)|^2 \tag{19}$$

$$\begin{aligned} &\gamma_{3e} + \gamma_{3b} A(z_1) \dots A(z_{k-1}) A((z_1 \dots z_{k-1})^{-1}) = \\ &\gamma_{3g} B(z_1) \dots B(z_{k-1}) B((z_1 \dots z_{k-1})^{-1}) \end{aligned} \tag{20}$$

Eq. (20) can be written :

$$\begin{aligned} &\gamma_{3e} + \gamma_{3b} \sum_{k,l,m=0}^{p} a_k a_l a_m z_1^{m-k} z_2^{m-l} = \\ &\gamma_{3g} \sum_{k,l,m=0}^{p} b_k b_l b_m z_1^{m-k} z_2^{m-l} \end{aligned} \tag{21}$$

with $a_0 = b_0 = 1, a_i = 0$ for $|i| > p$ and $b_j = 0$ for $|j| > p$.

For $m - k = m - l = p$,

$$\gamma_{3b} \sum_{m=0}^{p} a_{m-p}^2 a_m z_1^p z_2^p = \gamma_{3g} \sum_{m=0}^{p} b_{m-p}^2 b_m z_1^p z_2^p. \tag{22}$$

Hence,

$$\gamma_{3b} a_0^2 a_p = \gamma_{3g} b_0^2 b_p \Longleftrightarrow \gamma_{3b} a_p = \gamma_{3g} b_p \tag{23}$$

For $m - k = p$ **and** $m - l = i \in \{1, ..., p-1\}$, Eq. (32) leads similarly to:

$$\gamma_{3b} a_0 a_i a_p = \gamma_{3g} b_0 b_i b_p \Longleftrightarrow \gamma_{3b} a_i a_p = \gamma_{3g} b_i b_p \tag{24}$$

With the assumption $\gamma_{3b} \neq 0$, Eq. (23) and (24) lead to $\gamma_{3g} \neq 0$ and:

$$b_i = a_i \qquad i \in \{1, ..., p-1\} \tag{25}$$

Thus, the polynomial $B(z)$ can be expressed as:

$$B(z) = \sum_{k=0}^{p-1} a_k z^{-k} + \frac{\gamma_{3b}}{\gamma_{3g}} a_p z^{-p}. \tag{26}$$

For $m - k = p$ **and** $m - l = 0$, Eq. (32) yields:

$$\gamma_{3b} a_0 a_p^2 = \gamma_{3g} b_0 b_p^2 \Longleftrightarrow \gamma_{3b} a_p^2 = \gamma_{3g} b_p^2 \tag{27}$$

Comparing Eq. (23) and Eq. (27) leads to $a_p = b_p$ (i.e. $B(z) = A(z)$) and $\gamma_{3b} = \gamma_{3g}$. Eq. (19) can be rewritten as:

$$\gamma_{2e} + (\gamma_{2b} - \gamma_{2g}) |A(z)|^2 = 0 \tag{28}$$

It follows easily that Eq. (28) cannot be satisfied when $A(z)$ is the Z transform of a pth order AR process. A similar result can be obtained when $\gamma_{3b} = 0$, provided that $b(n)$ and $e(n)$ are not simultaneously Gaussian.

7 Appendix B : Third Order Cumulants of $y_1(n)$ and $y_2(n)$.

This appendix presents theoretical expressions for the third order cumulants of the noisy AR(1) process $y_1(n)$ (parameters $[1; a_1]$) and its SE ARMA process $y_2(n)$.

Noisy AR Process

$$C_{y1}(\tau_1, \tau_2) = \frac{\gamma_{3e}}{1 + a_1^3} (-a_1)^{\tau_1 + \tau_2} \tag{29}$$

Spectrally Equivalent ARMA Process

$\boldsymbol{\tau}_1 \neq \mathbf{0}, \boldsymbol{\tau}_2 \neq \mathbf{0}$

$$C_{y2}(\tau_1, \tau_2) = \frac{\gamma_{3g}}{1 + a_1^3} (-a_1)^{\tau_1 + \tau_2} f_1(a_1, \gamma_{2e}, \gamma_{2b}) \tag{30}$$

with:

$$f_1(a_1, \gamma_{2e}, \gamma_{2b}) = \gamma^2 \left[1 + a_1^3 (1 - \gamma)\right] \tag{31}$$

and:

$$\gamma = 1 - \frac{\gamma_{2b}}{\gamma_{2g}}$$

$\boldsymbol{\tau}_1 \neq \mathbf{0}, \boldsymbol{\tau}_2 = \mathbf{0}$

$$C_{y2}(\tau_1, 0) = \frac{\gamma_{3g}}{1 + a_1^3} (-a_1)^{\tau_1} f_2(a_1, \gamma_{2e}, \gamma_{2b}) \tag{32}$$

with:

$$f_2(a_1, \gamma_{2e}, \gamma_{2b}) = \gamma \left[1 + a_1^3 \left(1 - \gamma^2\right)\right] \tag{33}$$

The case $\boldsymbol{\tau}_2 \neq \mathbf{0}, \boldsymbol{\tau}_1 = \mathbf{0}$ is obtained by replacing τ_1 by τ_2 in Eq. (32).

$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \mathbf{0}$

$$C_{y2}(\tau_1, \tau_2) = \frac{\gamma_{3g}}{1 + a_1^3} f_3(a_1, \gamma_{2e}, \gamma_{2b}) \tag{34}$$

with:

$$f_3(a_1, \gamma_{2e}, \gamma_{2b}) = 1 + a_1^3 \left(1 - \gamma^3\right) \tag{35}$$

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