ARMA Parameter Estimation Through Enhanced Double MA Modelling

Achilleas G. Stogioglou and Stephen McLaughlin
Signals and Systems Group, Department of Electrical Engineering ,
The University of Edinburgh

ABSTRACT

This paper considers the application of MA cumulant enhancement to the identification of the parameters of a causal nonminimum phase ARMA(p,q) system which is excited by an unobservable independent identically distributed (IID) non-Gaussian process. The method proposed in this paper is based on the double MA method of [1]. The cumulant enhancement is used to improve the cumulants of the two intermediate MA models which result from the decomposition of the original ARMA(p,q) model. Simulation results are presented to demonstrate the effects of cumulant enhancement on the estimated ARMA parameters.

1 Introduction

Consider a real stationary random process $\{y(n)\}$ satisfying the following differnce equation:

$$\sum_{i=0}^{p} \alpha(i)x(k-i) = \sum_{i=0}^{q} b(i)w(k-i), \tag{1}$$

$$y(n) = x(n) + v(n). \tag{2}$$

where $\{w(n)\}$ is an unobservable, stationary, zero-mean, IID, non-Gaussian process and $v(n)$ is an additive noise process which is independant of the $x(n)$ and is assumed to be zero-mean Gaussian and perhaps coloured. This paper deals with the problem of estimating the ARMA model parameters $a(i), i = 0, ..., p$ and $b(i), i = 0, ..., q$ from just the noisy observations of the output process $y(n)$.

A number of different linear methods to solve this problem have been reported in the literature [1, 2, 3]. The method of [1] is particularly interesting because it decomposes the problem of ARMA parameter estimation to two MA parameter estimation problems. In this paper we follow the method of [1], which we refer to as the double MA method. After summarising the main steps of the double MA algorithm, this paper considers the enhancement of the cumulants of the two MA models [4] that result from the decomposition of the original ARMA parameter estimation problem. The enhanced cumulants can then be used for the estimation of the parameters of the two MA models.

2 The Double MA Method

The transfer function corresponding to model (1) is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{q} b(i)z^{-i}}{\sum_{i=0}^{p} \alpha(i)z^{-i}}. \tag{3}$$

The z-transform of the third-order cumulants of $\{y(n)\}$ are given by the following equation [5]:

$$\begin{aligned} C_{3,y}(z_1, z_2) = \gamma_{3,x} H(z_1)H(z_2)H(z_1^{-1}z_2^{-2}) = \\ \gamma_{3,x}\frac{B(z_1)B(z_2)B(z_1^{-1}z_2^{-1})}{A(z_1)A(z_2)A(z_1^{-1}z_2^{-1})}. \end{aligned} \tag{4}$$

In [1], it was shown that the problem of estimating the ARMA(p,q) parameters can be reduced to two MA estimation problems. According to [1], equation (4) can be written as

$$\begin{aligned} C_{3,y}(z_1, z_2)A(z_1)A(z_2)A(z_1^{-1}z_2^{-1}) = \\ \gamma_{3,x}B(z_1)B(z_2)B(z_1^{-1}z_2^{-1}). \end{aligned} \tag{5}$$

In the time domain equation (5) becomes

$$\sum_{i,j=-p}^{p} \alpha_3(i,j)c_{3,y}(m-i, n-j) = \begin{cases} 0 & m,n \notin S(q) \\ \gamma_{3,x}b_3(m,n) & m,n \in S(q), \end{cases} \tag{6}$$

where $S(q)$ is the finite domain of support of the third-order cumulants of MA(q) processes. As reported in [1], equation (6) can be used to estimate the coefficients $\alpha_3(i,j)$ and $b_3(i,j)$ which are then considered as third-order cumulants of MA models corresponding to the AR part and the MA part of the ARMA model respectively. At this stage we make use of the MA cumulant enhancement method of [4] to obtain enhanced sets of cumulants $\alpha_{3,e}(i,j)$ and $b_{3,e}(i,j)$ which can then be used for parameter estimation.

3 MA Cumulant Enhancement

The error in the estimation of cumulants involved in equation (6), results in error in the estimated coefficients $\alpha_3(i,j)$ and $b_3(i,j)$. However we know that in theory these coefficients can be regarded as cumulants of some MA(p) and MA(q) processes respectively. The theoretical properties of the coefficients, can be used by the MA cumulant enhancement method [4], to reduce the error of the estimated coefficients.

The MA cumulant enhancement method is an iterative method based on the concept of Composite Property Mappings [6, 7]. Here we briefly describe the algorithm for the enhancement of the coefficients $\alpha_3(i,j)$. The procedure for the coefficients $b_3(i,j)$ is identical. We use the estimated coefficients $\alpha_3(i,j)$ to form the following vectors:

$$\mathbf{a_n} = [\alpha_3(n-p,n), \alpha_3(n-p+1,n), \ldots, \alpha_3(p-1,n), \alpha_3(p,n), \underbrace{0,\ldots,0}_{n}]^\top. \quad (7)$$

Also, for $0 \leq d \leq p$ we define as $\mathbf{a_p^d}$ the following vector:

$$\mathbf{a_p^d} = [\underbrace{0,\ldots,0}_{d}, \alpha_3(0,p), \alpha_3(1,p), \alpha_3(2,p), \ldots, \alpha_3(p,p), \underbrace{0,\ldots,0}_{p-d}]^\top. \quad (8)$$

The vectors $\mathbf{a_p^d}$ and $\mathbf{a_n}$ have $2p+1$ elements each. Finally, the vectors defined by equations 7 and 8 are used to construct the following matrix:

$$\mathbf{A_{3,p}} = [\mathbf{a_p}, \mathbf{a_p^1}, \mathbf{a_p^2}, \ldots, \mathbf{a_p^p}, \mathbf{a_{p-1}}, \mathbf{a_{p-1}}, \ldots, \mathbf{a_1}, \mathbf{a_0}]^\top. \quad (9)$$

$\mathbf{A_{3,p}}$ is a $(2p+1)\times(2p+1)$ matrix. The rank of $\mathbf{A_{3,p}}$ is $p+1$ (Rank Property) [4] and additionally it has a theoretical linear structure which is dictated by the theoretical symmetries of the coefficients $\alpha_3(i,j)$ (Structure Property).

With the use of SVD we can implement a mapping $\mathcal{F}_{p+1}$, which maps a full rank matrix $\mathbf{X}$ to the matrix $\mathcal{F}_{p+1}(\mathbf{X})$ which is the "nearest" matrix to $\mathbf{X}$ with rank $p+1$. More specifically, let $\mathbf{X} \in \mathbf{R}^{(2p+1)\times(2p+1)}$ whose SVD is given by,

$$\mathbf{X} = \sum_{k=1}^{2p+1} \sigma_k \mathbf{u}_k \mathbf{v}_k'. \quad (10)$$

Then (see [6, 8]) the property mapping can be defined as,

$$\mathcal{F}_{p+1}(\mathbf{X}) = \sum_{k=1}^{p+1} \sigma_k \mathbf{u}_k \mathbf{v}_k'. \quad (11)$$

For the second property, we seek a mapping $\mathcal{F}_\mathbf{A}$, that maps a given matrix $\mathbf{X}$ to the "nearest" matrix $\mathcal{F}_\mathbf{A}(\mathbf{X})$, which has a linear structure characterised Before presenting a mapping corresponding to the structure property it is instructive to formally define the structure of the matrix $\mathbf{A_{3,p}}$. Suppose T denotes a linear transformation from $\mathbf{R}^{(2p+1)\times(2p+1)}$ to $\mathbf{R}^{(2p+1)(2p+1)}$ such that if $\mathbf{x} = T(\mathbf{X})$ then $\mathbf{x}$ is the concatenation of column vectors of $\mathbf{X}$. Then there exists a $(2p+1)^2\times(p+1)(p+2)/2$ matrix $\mathbf{A}$ (called the *characteristic matrix*) such that $T(\mathbf{C}) = \mathbf{A}\theta$. The matrix $\mathbf{A}$ has rows which either have all their elements zero, or one element equal to one and the rest zero ($\mathbf{A}$ is a sparse matrix). by matrix $\mathbf{A}$ [6, 4]. Such a mapping is given in [6] as,

$$\mathcal{F}_\mathbf{A}(\mathbf{X}) = T^{-1}(\mathbf{A}[\mathbf{A}'\mathbf{A}]^{-1}\mathbf{A}'\ T(\mathbf{X})). \quad (12)$$

We define the composite property mapping $\mathcal{F}$ as follows:

$$\mathcal{F} = \mathcal{F}_\mathbf{A}\mathcal{F}_{p+1}. \quad (13)$$

We can construct property mappings $\mathcal{F}_r$ and $\mathcal{F}_s$ corresponding to the two properties so that every $(2p+1)\times(2p+1)$ matrix is mapped to the "nearest" matrix possesing the desired property. Starting from the initial coefficient matrix $\mathbf{A_{3,p}}$ we obtain a sequence of matrices $\mathbf{A_{3,p}}^{(i)}$ according to the rule

$$\mathbf{A_{3,p}}^{(i+1)} = \mathcal{F}_s(\mathcal{F}_r(\mathbf{A_{3,p}}^{(i)})). \quad (14)$$

Assuming that the iterative algorithm converges the resulting matrix has been shown to consist of true cumulants of some MA(p) model. Even if the iterative algorithm is stopped before convergence has been achieved, the final matrix is closer to a matrix with both the prescribed properties.

After performing cumulant enhancement, on both $\alpha_3(i,j)$ and $b_3(i,j)$, any MA parameter estimation method can be used for the estimation of the ARMA parameters $a(i), i = 0, ..., p$ and $b(i), i = 0, ..., q$. In order to study the effect of cumulant enhancement on the estimated parameters, here we use the same approach with [1] for the estimation of the unknown parameters:

$$b(i) = \frac{2}{q(q+1)} \sum_{m=0}^{q} \sum_{n=0}^{m} \frac{b_3^{1/3}(m,n) b_3(q,j)}{B(m,n)} \quad (15)$$

where

$$B(m,n) = \sum_{k=0}^{q-m} b_3(q,k) b_3(q,k+m) b_3(q,k+n)^{1/3}. \quad (16)$$

The same method is used to estimate the $a(i), i = 0, ..., p$ with respect to the $\alpha_3(i,j)$.

4 Numerical Simulations

Monte Carlo simulations are provided to demonstrate the effect of MA cumulant enhancement on ARMA parameter estimation. The following signal models have been used in the simulations:

Signal model I

$$\begin{aligned} x(n) - 0.05x(n-1) - 0.6x(n-2) = \\ u(n) - 1.25u(n-1) \\ y(n) = x(n) + v(n) \end{aligned} \quad (17)$$

The poles of the system are 0.8 and -0.75. The zero of the system is 1.25. This is a causal non-minimum phase system.

Signal model II

$$\begin{aligned} x(n) - 2.2x(n-1) + 1.77x(n-2) \\ -0.520x(n-3) = u(n) - 1.250u(n-1) \\ y(n) = x(n) + v(n) \end{aligned} \quad (18)$$

The poles of the system are 0.8 and $0.7 \pm j0.4$ (modulus=0.806226). The zero of the system is 1.25. This is a non-minimum phase model with All-Pass factor.

Signal model I: 2000 samples		
Parameters	Double MA	Enhanced
$a(1) = -0.050$	0.061 ± 0.304	-0.080 ± 0.187
$a(2) = -0.600$	-0.788 ± 0.421	-0.834 ± 0.228
$b(1) = -1.250$	-1.027 ± 1.359	-1.267 ± 0.488

Table 1: Mean +/- Standard deviation averaged over 50 Monte Carlo runs. SNR=50dB's.

Signal model II: 4000 samples		
Parameters	Double MA	Enhanced
$a(1) = -2.200$	-1.558 ± 0.098	-1.955 ± 0.079
$a(2) = 1.770$	0.927 ± 0.148	1.521 ± 0.133
$a(3) = -0.520$	-0.171 ± 0.086	-0.377 ± 0.092
$b(1) = -1.250$	-2.442 ± 1.119	-1.112 ± 0.137

Table 2: Mean +/- Standard deviation averaged over 50 Monte Carlo runs. SNR=50dB's.

The driving sequence $u(n)$ follows a zero-mean, IID exponentially distributed process with $c_{2,u}(0) = 1$ and $c_{3,u}(0,0) = 2$. The additive noise process $v(n)$ is assumed to be white, Gaussian and independent of $x(n)$. The signal to noise ratio is 50dB's. The results are averaged over 50 Monte Carlo simulations.

Simulation results for the non-minimum phase signal model I are presented in table 1. The number of output samples in each Monte Carlo run is 2000. The table shows that use of enhanced cumulants for the estimation of the AR parameters reduces the variance of the estimates. In the case of the MA parameter the improvement obtained after cumulant enhancement is more significant both in bias and variance.

Results for the signal model II are presented in table 2. In this case the number of output samples in each Monte Carlo run is 4000. The AR parameters obtained after cumulant enhancement have lower variance and are less biased than those obtained without cumulant enhancement. Once again the improvement is more significant in the case of the MA parameter.

5 Conclusions

This paper considered the problem of ARMA parameter estimation using only third-order cumulants of the output process. The double MA method [1] for ARMA parameter estimation decomposes the estimation problem into two MA estimation sub-problems. In this paper we considered the application of MA cumulant enhancement to remove the estimation error from the cumulants corresponding to the MA and AR part of the ARMA model. Numerical results were presented showing that the introduction of cumulant enhancemnet into the double MA method can result in more accurate ARMA parameters.

References

[1] G B Giannakis and A Swami. On estimating noncausal nonminimum phase arma models of non-gaussian processes. *IEEE Transactions on ASSP*, 38(3):478–495, March 1990.

[2] A Swami and J Mendel. ARMA Parameter Estimation Using Only Output Cumulants. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 38(7):1257–1265, July 1990.

[3] D Zazula J-L Vuattoux and E Le Carpentier. Causal ARMA Parameter Estimation Using a Linear Combination of Third-Order Cumulant Slices. In *Proceeding of IEEE Signal Processing ATHOS Workshop on Higher-Order Statistics*, pages 464–468, Girona, Spain, June 1995.

[4] A G Stogioglou, S McLaughlin. MA Cumulant Enhancement and Parameter Estimation. *IEEE Transaction on Signal Processing*, July 1996.

[5] D Brillinger and M Rosenblatt. Computation and Iterpretation of k-th Order Spectra. In B Harris, editor, *Spectral Analysis of Time Signals*, pages 907–938. Wiley, 1967.

[6] J Cadzow. Signal Enhancement- A Composite Property Mapping Algorithm. *IEEE Transactions on Acoustics, Speech and Signal Preocessing*, 36(1):49–62, 1988.

[7] P L Combettes. The Foundations of set Theoretic Estimation. *Proceedings of the IEEE*, 81(2):181208, February 1993.

[8] G Golub and C Van Loan. *Matrix Computations*. The Johns Hopkins University Press, 1983.