DESIGNING OF ROBUST STABLE DIGITAL FILTERS

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ABSTRACT

The Ackerman-Barmish method was used to establish a set of *stable family* of an Infinite Impulse Response (IIR) digital filters. Next, the optimization method was used to choose a filter which meets design specifications given in the frequency domain. Designing of lowpass third order IIR filter is presented as an example.

1 INTRODUCTION

The recursive equation which combines the input x_n and the output y_n of IIR filter is given by

$$y_n = \sum_{i=0}^{N} a_i x_{n-i} - \sum_{i=1}^{N} b_i y_{n-i} \qquad \text{for} \quad n = 1, 2, \ldots \tag{1}$$

where N is the order of filter and a_i, b_i are constant coefficients. Both signals, input and output, are scalar functions of discrete time denoted by n.

Designing an IIR filter consists in determining a stable computational procedure (1) that meets design specifications. These specifications are typically given in the frequency domain. The feedback loop in IIR filter complicates the designing procedure. The goal of this paper is to present the method for finding such set $B \subset \Re^N$ that filter (1) is stable for all coefficients $b = (b_1, b_2, \ldots, b_N) \in B$. The optimization method is next used to choose from the set *B* such filter (1) which meets design specifications in a best way.

2 ROBUST STABILITY

A discrete-time filter (1) is stable [2] if a bounded input sequence produces a bounded output sequence. It is well known that an IIR filter is asymptotic stable, if and only if all zeroes of its characteristic polynomial lie inside the unite circle. The digital filter (1) is said to be robust stable if it is defined a set $B \subset \Re^N$, such that each filter with coefficients $b = (b_1, b_2, \ldots, b_N) \in B$ is asymptotic stable.

Let us assume that we are given two vectors

$$\begin{aligned} \underline{b} &= (\underline{b_1}, \underline{b_2}, \ldots, \underline{b_N}) \in \Re^N \\ \overline{b} &= (\overline{b_1}, \overline{b_2}, \ldots, \overline{b_N}) \in \Re^N \end{aligned} \tag{2}$$

where $\underline{b_i} \le \overline{b_i}$ and i = 1, 2,..., N. Following Ackerman and Barmish [1], let us define two sets

$$B = \{ b \in \Re^N : \underline{b_i} \le b_i \le \overline{b_i};\ i = 1, \ldots, N \} \tag{3}$$

$$B_e = \{ b \in B : b_i = \underline{b_i} \cup b_i = \overline{b_i};\ i = 1, .., N \} \tag{4}$$

where *B is* the rectangular prism (3) with vertices defined by (4). The problem is to ascertain whether IIR filters remain asymptotic stable for all *b* from a bounded set *B*. For filter (1) let us define a set of characteristic polynomials

$$P_j(z) = z^N + \sum_{i=1}^{N} b_i^j z^{N-i} \qquad \text{for } j = 1, \ldots, 2^N \tag{5}$$

where $b^j \in B_e$ denotes the *j*-th extreme point of *B*. For the polynomial P let us define (N-1)x(N-1) matrix

$$S(P) = S_U(P) - S_L(P) \tag{6}$$

where

$$S_U(P) = \begin{bmatrix} 1 & b_1 & b_2 & \cdots & b_{N-3} & b_{N-2} \\ 0 & 1 & b_1 & \cdots & b_{N-4} & b_{N-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$S_L(P) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & b_N \\ 0 & 0 & 0 & \cdots & b_N & b_{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_N & b_{N-1} & \cdots & b_4 & b_3 \\ b_N & b_{N-1} & b_{N-2} & \cdots & b_3 & b_2 \end{bmatrix}$$

Theorem 1 [1]:

If all generating polynomials

$$\left\{P_j(z) : j = 1,2,\ldots,2^N\right\} \tag{7}$$

have zeros inside the unite circle and for all $i > j$ such that $i, j \in \{1,2,...,2^N\}$ the matrices $S(P_i)S^{-1}(P_j)$ have no real eigenvalues in $(-\infty, 0)$ then and only then all polynomials $P(\cdot) \in \mathbf{P}$ have their zeros inside the unite circle. ■

The above Ackerman-Barmish theorem can be used for verifying, in a finite number of steps, the stability of the family of filters. The other important problem consists in finding the largest possible set B. This task is solved in an iterative way. At the beginning of calculations it is assumed that the initial set B_0 consists of only one point and

$$\underline{b_i} = \overline{b_i} = 0 \quad \text{for} \quad i = 1,...,N \tag{8}$$

It means that we start from a FIR filters. Next, the sequence of closed increasing subsets

$$\{0\} = B_0 \subset B_1 \subset \cdots \subset B_j \subset \cdots \subset B \subset \Re^N \tag{9}$$

is generated. The sets B_j for j=1,2,... are determined by the edge values of polynomial coefficients (2). There is no unique method for constructing the sequence (9). The increase of their magnitudes can be made by succeeding substitutions

$$\begin{aligned} \overline{b_i} &\Leftarrow \overline{b_i} + \Delta_i \\ \underline{b_i} &\Leftarrow \underline{b_i} - \Delta_i \end{aligned} \tag{10}$$

where $\Delta_i > 0$ are assumed speeds of searching. Each substitution must be verified and only substitution which generates the robust stable set can be accepted. B_j are subsets which have forms of parallelepiped inscribed in an irregular set of robust stable filter coefficients. The final set B depends on the magnitudes of Δ_j and the number of iterations.

3 FILTER OPTIMIZATION

The Powell optimization method can be used to find the optimal parameters a_i , b_i for filter (1). The quality criterion can be taken in the form

$$Q = \int_0^{f_{\max}} \left| H(f) - A(f)\, e^{j\theta(f)} \right|^2 df \tag{11}$$

where $H(f)$ is a transfer function of filter (1). $A(f)$ and $\theta(f)$ are assumed amplitude and phase characteristics, respectively.

4 EXAMPLE

Let us consider an IIR filter of third order described by the recursive equation

$$\begin{aligned} y_n = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + a_3 x_{n-3} - \\ b_1 y_{n-1} - b_2 y_{n-2} - b_3 y_{n-3} \end{aligned} \tag{12}$$

Our first task is to find "the largest possible" set B defined by (3). It means that we are looking for such boundary values of parameters b_i that filter (12) remain stable if its coefficients satisfy inequalities

$$\underline{b_i} \le b_i \le \overline{b_i} \quad \text{for} \quad i = 1, 2, 3 \tag{13}$$

For the considered example we have eight generating polynomials (5) and eight matrices $S(P_j)$. Under assumption

$$\Delta_i = 0.1 \quad \text{for} \quad i = 1, 2, 3 \tag{14}$$

the computer calculations showed that filter (12) remains stable if

$$\begin{aligned} -0.3 \le b_1 \le 0.3 \\ -0.3 \le b_2 \le 0.8 \\ -0.3 \le b_3 \le 0.3 \end{aligned} \tag{15}$$

For the considered problem the polytope of polynomials has generating polynomials (5) in the following forms

$$\begin{aligned} P_1(z) &= z^3 - 0.3z^2 - 0.3z - 0.3 \\ P_2(z) &= z^3 + 0.3z^2 - 0.3z - 0.3 \\ P_3(z) &= z^3 - 0.3z^2 + 0.8z - 0.3 \\ P_4(z) &= z^3 + 0.3z^2 + 0.8z - 0.3 \\ P_5(z) &= z^3 - 0.3z^2 - 0.3z + 0.3 \\ P_6(z) &= z^3 + 0.3z^2 - 0.3z + 0.3 \\ P_7(z) &= z^3 - 0.3z^2 + 0.8z + 0.3 \\ P_8(z) &= z^3 + 0.3z^2 + 0.8z + 0.3 \end{aligned} \tag{16}$$

The first assumption of the Ackerman-Barmish theorem is satisfied because all these polynomials are Schur stable. According to formula (6), polynomials (16) are associated with the matrices

$$S(P_1)=\begin{bmatrix}1 & 0\\ -0.3 & 1.3\end{bmatrix}\quad S(P_2)=\begin{bmatrix}1 & 0.6\\ -0.3 & 1.3\end{bmatrix}$$
$$S(P_3)=\begin{bmatrix}1 & 0\\ -0.3 & 0.2\end{bmatrix}\quad S(P_4)=\begin{bmatrix}1 & 0.6\\ -0.3 & 0.2\end{bmatrix}$$
$$S(P_5)=\begin{bmatrix}1 & -0.6\\ 0.3 & 1.3\end{bmatrix}\quad S(P_6)=\begin{bmatrix}1 & 0\\ 0.3 & 1.3\end{bmatrix} \tag{17}$$
$$S(P_7)=\begin{bmatrix}1 & -0.6\\ 0.3 & 0.2\end{bmatrix}\quad S(P_8)=\begin{bmatrix}1 & 0\\ 0.3 & 0.2\end{bmatrix}$$

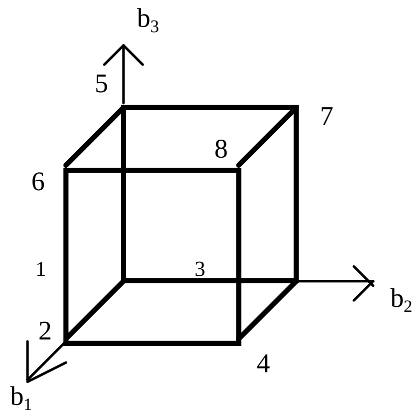


Fig.1 The exposed edges of the polytope and the rule of numbering the vertices.

There are twelve exposed edges (see Fig.1) of the polytope spanned by (16). For each exposed edge, according to the Ackerman-Barmish theorem, we calculate the eigenvalues λ_1, λ_2 of matrices $S(P_i)S^{-1}(P_j)$:

$$S(P_2)S^{-1}(P_1)=\begin{bmatrix}1.1385 & 0.4615\\ 0 & 1\end{bmatrix}\quad \lambda_1=1.1386,\ \lambda_2=1$$

$$S(P_3)S^{-1}(P_1)=\begin{bmatrix}1 & 0\\ -0.2538 & 0.1538\end{bmatrix}\quad \lambda_1=1,\ \lambda_2=0.1538$$

$$S(P_4)S^{-1}(P_2)=\begin{bmatrix}1 & 0\\ -0.223 & 0.2568\end{bmatrix}\quad \lambda_1=1,\ \lambda_2=0.2568$$

$$S(P_4)S^{-1}(P_3)=\begin{bmatrix}1.9 & 3\\ 0 & 1\end{bmatrix}\quad \lambda_1=1.9,\ \lambda_2=1$$

$$S(P_5)S^{-1}(P_1)=\begin{bmatrix}0.8615 & -0.4615\\ 0.6 & 1\end{bmatrix}$$
$$\lambda_{1,2}=0.9307\pm 0.5217i$$

$$S(P_6)S^{-1}(P_2)=\begin{bmatrix}0.8784 & -0.4054\\ 0.527 & 0.7568\end{bmatrix} \tag{18}$$
$$\lambda_{1,2}=0.8176\pm 0.4582i$$

$$S(P_6)S^{-1}(P_5)=\begin{bmatrix}0.8784 & 0.4054\\ 0 & 1\end{bmatrix}\quad \lambda_1=0.8784,\ \lambda_2=1$$

$$S(P_7)S^{-1}(P_3)=\begin{bmatrix}0.1 & -3\\ 0.6 & 1\end{bmatrix}\quad \lambda_{1,2}=0.55\pm 1.2639i$$

$$S(P_7)S^{-1}(P_5)=\begin{bmatrix}1 & 0\\ 0.223 & 0.2568\end{bmatrix}\quad \lambda_1=1,\ \lambda_2=0.2568$$

$$S(P_8)S^{-1}(P_4)=\begin{bmatrix}0.5263 & -1.5789\\ 0.3158 & 0.0527\end{bmatrix}$$
$$\lambda_{1,2}=0.2895\pm 0.6652i$$

$$S(P_8)S^{-1}(P_6)=\begin{bmatrix}1 & 0\\ 0.2538 & 0.1538\end{bmatrix}\quad \lambda_1=0.1538,\ \lambda_2=1$$

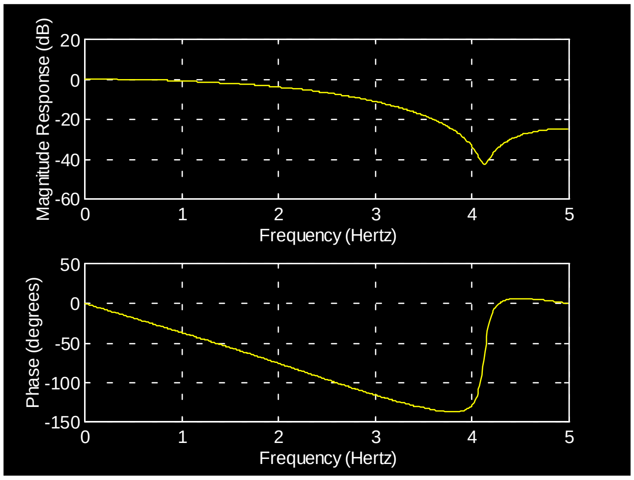
$$S(P_8)S^{-1}(P_7)=\begin{bmatrix}0.5263 & 1.5789\\ 0 & 1\end{bmatrix}\quad \lambda_1=0.5263,\ \lambda_2=1$$

There are no negative eigenvalues. Due to this the second assumption of the Ackerman-Barmish theorem is also satisfied.

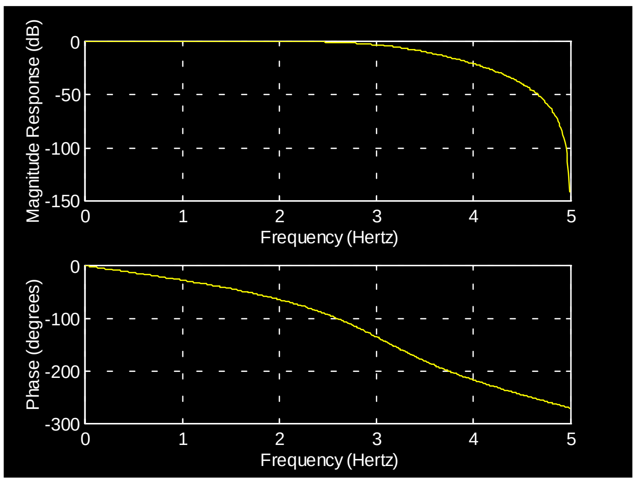
The next task is to compute the coefficients of filter (12). Let us assume that we intend to design a low pass filter with cutoff frequency 3000 Hz and the linear phase characteristic

$$\theta(f)=-360\, f\, \Delta t \tag{19}$$

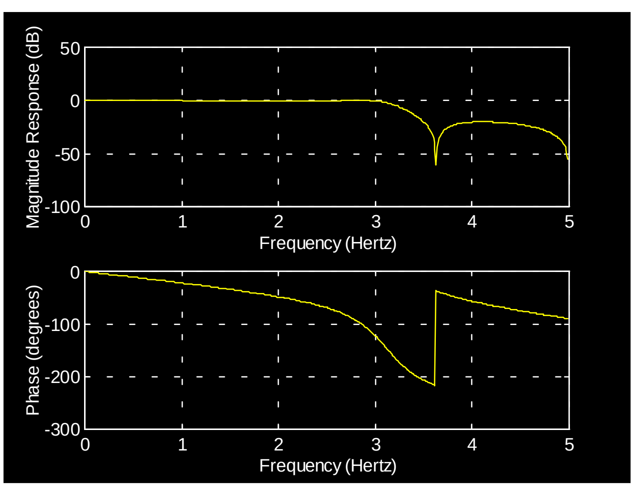
where $\Delta t = 0.0001$ is the sampling rate. The Nyquist frequency in (11) is thus equal to $f_{\max}=5000$ Hz. The computer calculations showed that criterion (11) has a minimum value for the coefficients presented in the second column of Tab.1. The filter is stable because its coefficients b_1, b_2, b_3 fulfil the inequality constraints (15). The characteristics of this filter are compared (see Fig.2) with the characteristics of Butterworth and elliptic filters. It is very interesting that an IIR filter designed by minimizing the quality criterion (11) has a linear phase characteristic in the passband.



Filter designed by applying the optimization method



Butterworth filter



Elliptic filter

Fig.2 Characteristics of the third-order IIR filters

Tab.1 The parameters of IIR digital filters.

	Optimization method	Butterworth	Elliptic
a_o	0.2574	0.2569	0.3651
a_1	0.4208	0.7707	0.8405
a_2	0.2349	0.7707	0.8405
a_3	- 0.0029	0.2569	0.3651
b_1	- 0.1735	0.5772	0.6676
b_2	0.0962	0.4218	0.7141
b_3	- 0.0236	0.0563	0.0296

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