ELIMINATION OF LIMIT CYCLES IN A DIRECT FORM DELTA OPERATOR FILTER

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ABSTRACT

Delta operator realizations have been found to be robust against roundoff errors when high sampling rate relative to signal bandwidth is used. In this paper zero input limit cycles in the transposed direct form delta operator structure are studied. It is shown that the limit cycles of the basic delta structure are much lower in amplitude than those of the direct form delay structure for narrowband lowpass filters. Moreover, by certain modifications to the delta operator the zero input limit cycles can be completely avoided. It is also shown that narrowband lowpass filters with both low roundoff noise and absence of limit cycles can be implemented.

1. INTRODUCTION

The good finite-wordlength performance of delta operator structures has received a lot of interest recently [1]. It has been reported that under fast sampling the delta realizations have superior roundoff noise and coefficient sensitivity properties over the traditional delay realizations. However, it has been found out that the delta realizations using fixed-point arithmetic may suffer from limit cycles [2]. In this paper, we study the transposed direct form II (DFIIt) delta operator structure, which is known to have excellent roundoff noise performance [3]. We apply the modified Tsypkin criterion to determine limit cycle free regions in the parameter space [4-6]. Moreover, we study amplitudes of limit cycles, when they exist, as absolute upper bounds.

It is shown that in the conventional delta operator structure absence of limit cycles cannot be guaranteed but, for systems having poles at low angles, limit cycles are strongly attenuated as compared to the corresponding delay realizations. We propose modifications into the delta operator which results in implementations free of zero input limit cycles when magnitude truncation quantizer is used. Complete absence of limit cycles can be guaranteed for any stable filter at the expense of increased roundoff noise at low pole angles. However, with a slightly more complex operator it is also possible to retain the good roundoff noise performance of the conventional delta structure for systems having poles close to $z = 1$ while limit cycles are still eliminated.

2. CRITERION FOR ABSENCE OF LIMIT CYCLES

For nonlinear feedback systems with only one nonlinearity there exist a powerful criterion due to Tsypkin which can be used to determine if a given system is free from autonomous zero-input oscillations (limit cycles) [4]. A general nonlinear system with one quantization nonlinearity is shown in Fig. 1. The Tsypkin criterion is based on a limited-gain quantizer which has a finite gain of value k at most. The quantizer satisfies a sector condition $y_0(n)[y_0(n) - kx_0(n)] \le 0$, where $k = 1$ for magnitude truncation and $k = 2$ for rounding, see Fig. 2 [5,6]. The quantization is modeled as an additive error

$$y_0(n) = x_0(n) + e(n). \tag{1}$$

Z-transforming (1) and solving the relation between the quantizer output and error, we obtain

$$Y_0(z) = \frac{1}{1-W(z)} E(z) = \frac{B(z)}{A(z)} E(z), \tag{2}$$

where $W(z) = X_0(z)/Y_0(z)$ is the linear part of the system. It can be shown [6] that autonomous oscillations under zero-input condition are not possible if

$$k \,\mathrm{Re}\left\{\frac{B(z)}{A(z)}\right\} + (1-k)\left|\frac{B(z)}{A(z)}\right|^2 > 0 \qquad \forall\ z = e^{j\omega}. \tag{3}$$

In this form the criterion is difficult to use, and it has been reformulated for second-order sections in terms of the filter coefficients in [6]:

$$P(x) = \frac{p_0 + p_1 x + p_2 x^2}{(1+x)^2} > 0 \qquad \forall x \ge 0, \tag{4}$$

where for magnitude truncation $p_0 = (1 + a_1 + a_2)(1 + b_1 + b_2)$, $p_1 = 2[1 + a_1 b_1 + a_2 b_2 - 3(a_2 + b_2)]$, $p_2 = (1 - a_1 + a_2)(1 - b_1 + b_2)$ and a_i and b_i are the coefficients of $A(z)$ and $B(z)$ respectively. In this text we consider only magnitude truncation for quantization. The inequality in (4) is satisfied if

$$p_0 > 0, \quad p_2 > 0, \quad p_1 > -2\sqrt{p_0 p_2}. \tag{5}$$

The criterion in (4) was obtained by converting (3) into the s-domain using the *bilinear transform* [8]. The variable x is squared s-domain frequency variable, so the relation be-

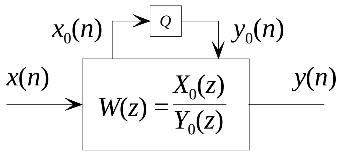


Fig. 1. *A general nonlinear feedback system with a single quantizer.*

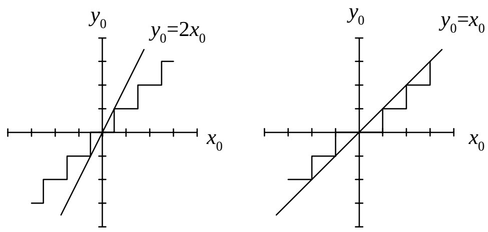


Fig. 2. *A sector condition for rounding and magnitude truncation quantizers.*

tween x and the z-domain frequency variable ω is

$$x = \tan^2(\omega/2)\ . \tag{6}$$

In particular, $\omega = 0$ is mapped to $x = 0$ and $\omega = \pm\pi$ are mapped to $x = \infty$.

3. LIMIT CYCLES IN DELTA REALIZATIONS

The delta DFIIt structure is shown in Fig. 3. We assume that double precision internal arithmetic is used and only one quantizer is included in the feedback loop. This means that other possible quantizers in the delta operators etc. must be implemented with increased precision so that they are not observable at the output. Next, we introduce the modified delta operator, which is defined as:

$$\delta_i = \frac{z - \rho_i}{\Delta_i}\ , \tag{7}$$

where $\rho_i \in$ [-1,1] and $\Delta_i \in$ (0,1]. The delta domain transfer function is

$$H_\delta(\delta_1, \delta_2) = g\,\frac{\beta_0 + \beta_1\delta_1^{-1} + \beta_2\delta_1^{-1}\delta_2^{-1}}{1 + \alpha_1\delta_1^{-1} + \alpha_2\delta_1^{-1}\delta_2^{-1}}\ , \tag{8}$$

where g is introduced to have unity passband gain. Coefficients of this parametrization are related to z-domain transfer function $H(z) = gN(z)/D(z)$ as: $\beta_0 = n_0$, $\beta_1 = (n_0(\rho_1 + \rho_2) + n_1)/\Delta_1$, $\beta_2 = (n_0\rho_2^2 + n_1\rho_2 + n_2)/(\Delta_1\Delta_2)$, $\alpha_1 = (\rho_1 + \rho_2 + d_1)/\Delta_1$, $\alpha_2 = (\rho_2^2 + d_1\rho_2 + d_2)/(\Delta_1\Delta_2)$. Note that $A(z) = D(z)$ but $B(z)$ is usually different from $N(z)$. For the first order system $\alpha_2 = \beta_2 = \rho_2 = 0$. In this text, we use $\Delta_i = 1$ for all i. Solving for the $B(z)/A(z)$ we obtain:

$$\frac{B(z)}{A(z)} = \frac{1 - \rho_1 z^{-1}}{1 + a_1 z^{-1}}\ , \tag{9}$$

for the first-order systems and

$$\frac{B(z)}{A(z)} = \frac{1 - (\rho_1 + \rho_2)z^{-1} + \rho_1\rho_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}\ , \tag{10}$$

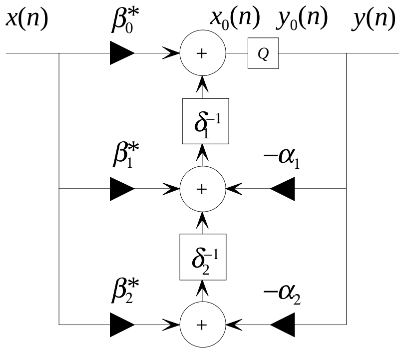


Fig. 3. *The delta DFIIt structure. Asterisk (*) in the coefficients means that scaling is embedded.*

for the second-order systems.

3.1 First-Order Systems

For the first-order systems we consider only the case $\rho_1 = 1$. The criterion in (4) reduces to:

$$P(x) = \frac{p_0 + p_1 x}{1 + x} > 0 \qquad \forall x \geq 0\ , \tag{11}$$

which is satisfied when

$$p_0 = (1 + a_1)(1 + b_1) > 0, \quad p_1 = 2(1 + a_1 b_1) > 0\ . \tag{12}$$

By substituting $b_1 = -1$ into the equations in (9) it is seen that $p_0 = 0$ and $p_1 > 0$ for all stable filters and criterion is not strictly met, but is equal to zero when $x = 0$. However, criterion (3) can be allowed to have a zero value at distinct frequency point and $x = 0$ corresponds to the single frequency $\omega = 0$ [8]. In this case, in principle it is possible that DC-limit cycle could occur, but there is a zero in the error transfer function (9) at $z = 1$ and this limit cycle is filtered out. When magnitude truncation is used, the first-order delta DFIIt structure is free of limit cycles for all stable filters.

3.2 Second-Order Systems

3.2.1 Conventional Delta Operator ($\rho_1 = \rho_2 = 1$)

Substituting ρ_1 and ρ_2 into (10) the numerator coefficients are found to be $b_1 = -2$ and $b_2 = 1$. It follows that $p_0 = 0$ and $p_1 = -4(1 + a_1 + a_2)$ which is negative for a stable filter and the criterion is not satisfied. The absence of limit cycles cannot be guaranteed. However, when the poles of the system are close to $z = 1$, a double zero at $z = 1$ in the error transfer function strongly attenuates the amplitude of the limit cycles. In Fig. 4. upper bounds for the limit cycle amplitudes in second order DFIIt realizations are shown [7]. It is assumed that also in the delay realization only one quantization exists. For small pole angles the delta realization is superior to the delay realization.

3.2.2 Limit Cycle Free Configuration ($\rho_1 = -1$, $\rho_2 = 1$)

In this case the numerator coefficients of (10) are $b_1 = 0$ and $b_2 = -1$. Substituting these into (5) it is seen that $p_0 = p_2 = 0$

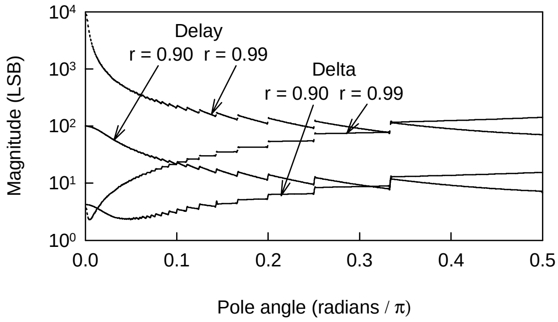


Fig. 4. *Upper bounds for limit cycle amplitudes of second order DFIIt sections as a function of the pole angle (note the scale), r is the pole radius.*

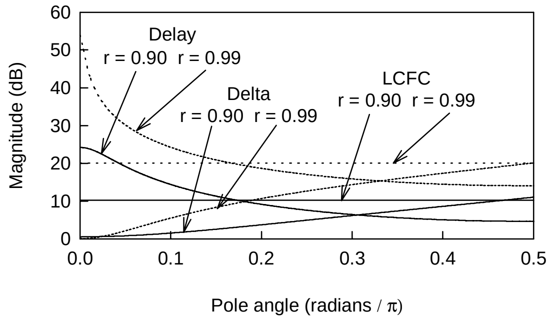


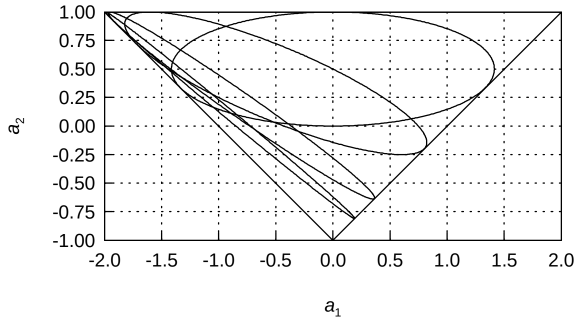
Fig. 5. *Noise gains of second order DFIIt structures when L_∞-scaling is used. Only one quantization assumed also in delay structure.*

and $p_1 = 8(1 - a_2)$, which is always positive for a stable filter. From (4) it is seen that the criterion polynomial $P(x)$ is equal to zero when $x = 0$ or $x = \infty$, so in principle limit cycles with angular frequencies $\omega = 0$ or $\omega = \pi$ are possible. However, the error transfer function has a zero at both the points $z = \pm 1$. Thus, when magnitude truncation is used, the absence of the zero-input limit cycles is guaranteed, see also [5,6]. The price to pay is the increased roundoff noise gain at small pole angles (LCFC in Fig 5). It is assumed that two's complement overflow characteristics can be used and L_∞-scaling is equivalent to embedding the gain of the transfer function into the numerator coefficients.

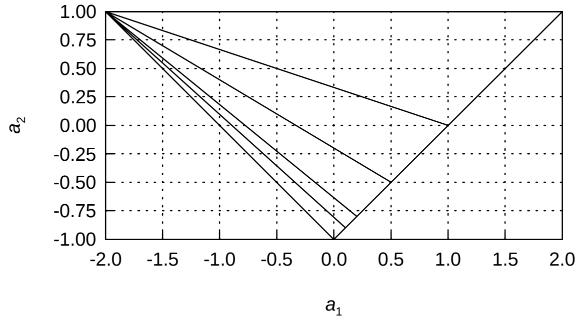
3.2.3 Partially Limit Cycle Free Configurations

Systems which are limit cycle free in some part of the parameter space can be obtained by choosing ρ_i:s to be smaller than one in absolute value. In the case of magnitude truncation limit cycle free region in the parameter (a_1, a_2)-space is limited by a stability triangle and by an ellipse (in some cases a line) inside this triangle [8]. If both the ρ_i are close to one, the limit cycle free region is very narrow. On the other hand, if a very small or negative ρ_i is used, roundoff noise increases for low pole angles, compare to Chapter 3.2.2. Because the delta operator structures are considered for the oversampled systems (poles are close to $z = 1$), we limit the ρ_i to be nonnegative.

When multiplying by ρ smaller than one in absolute



a)



b)

Fig. 6. *Limit-cycle-free regions for configurations 1 (a) and 2 (b). The parameter ρ = 0, 0.5, 0.8, and 0.9 (top to bottom). The LC-free region consists of the interior of the ellipse (which reduces to a line segment in b)) and the region of the triangle below it.*

value, quantization is needed also in the operator. If this quantization is done by magnitude truncation, the state variables (summations in the operators) converge to zero and internal limit cycles are avoided (if Tsypkin criterion holds). When rounding or two's complement truncation is used, it may happen that internal limit cycles occurs in the operator when the output of the system goes to zero. These so called hidden limit cycles can be allowed if they are absent at the output. Detailed study of this topic is beyond the scope of this text.

Multiplication by ρ is done for the double precision number and it may not be practical to implement it as a full precision multiplier. Parameter ρ could be rounded to power-of-two or one minus power-of-two, which result in simple implementations. Anyway, these configurations are more complex to implement than the conventional delta structure.

3.2.3.1 Configuration 1 ($\rho_1 = \rho_2 = \rho < 1$)

The limit cycle free regions for few values of ρ are shown in Fig. 6 a). The choice $\rho = 0$ corresponds to a delay realization. To solve value of ρ for a limit cycle free implementation $b_1 = -2\rho$ and $b_2 = \rho^2$ are substituted into the third inequality in (5). Note that with these values two first inequalities in (5) are satisfied for all the stable filters. After some algebra the following inequality is obtained:

$$V(\rho) = v_0 + v_1\rho + v_2\rho^2 < 0 \ , \qquad (13)$$

where $v_0 = 3a_2 - 1 - \sqrt{((1 + a_2)^2 - a_1^2)}$, $v_1 = 2a_1$ and $v_2 = 3 - a_2 + \sqrt{((1 + a_2)^2 - a_1^2)}$. It is seen that for stable filters v_2 is positive (and real) and $V(\rho)$ is a parabola opening upward. As a result the inequality in (13) is satisfied when ρ belongs to the open interval between the roots of $V(\rho)$ ($v_1^2 > 4v_0v_2$ for stable filters and $V(\rho)$ has distinct real roots).

3.2.3.2. Configuration 2 ($\rho_1 = \rho \le 1$, $\rho_2 = 1$)

The region where the absence of limit cycles can be guaranteed is shown in Fig. 6 b). Here, the choice of $\rho = 0$ corresponds to a system where one operator is the conventional delta operator and another one is a delay. By substituting $b_1 = -(\rho + 1)$ and $b_2 = \rho$ into (5) the following relation is obtained for a limit-cycle-free implementation:

$$\rho < \frac{1 - a_1 - 3a_2}{3 + a_1 - a_2}. \qquad (14)$$

The first inequality in (5) is equal to zero and criterion is not strictly met. This means that $P(x)$ in (4) is zero when $x = 0$ corresponding angular frequency $\omega = 0$. However, there is a zero at $z = 1$ in the error transfer function (9) and a DC limit cycle is not present in the output. The configuration 2 is interesting, because it is simpler to implement than configuration 1, but, as will be seen in the examples, the roundoff noise performance may not be very low for all the pole locations.

4. EXAMPLE

As an example sections of fourth-order elliptic lowpass filter are used with the numerator and denominator of the first section: $N_1(z) = 1 - 1.895z^{-1} + z^{-2}$ and $D_1(z) = 1 - 1.914z^{-1} + 0.950z^{-2}$, respectively, and those of the second section: $N_2(z) = 1 - 1.530z^{-1} + z^{-2}$ and $D_2(z) = 1 - 1.828z^{-1} + 0.843z^{-2}$. Table 1 shows that the conventional delta structure is superior to the delay structure in both the limit cycle and roundoff noise performance. Moreover, limit cycles can be eliminated using the configuration of Chapter 3.2.2, but roundoff noise gain is increased considerably, when compared to the conventional delta structure. Absolute upper bounds (AUB) for limit cycles are given in least significant bits and noise gain (NG) in decibels. In partially LC-free configurations ρ was chosen to be simply implementable (Table 2). Quantizations in the operators are to double precision and roundoff noise due to them is neglected. The roundoff noise performance of Configuration 1 is as good as with the conventional delta structure. Configuration 2 is not quite as good for the first section having larger pole angle and radius, but with the second section the noise gain is small.

Table 1.
Absolute upper bounds (AUB) and noise gains (NG) for delay, basic delta , and limit cycle free (LCFC) delta structure for the sections of the example filter.

Conf	Delay		Delta		LCFC	
Sec	1	2	1	2	1	2
AUB	136	76	2	6	-	-
NG	24.4	23.4	1.51	0.57	16.1	11.0

Table 2.
Results for partially LC-free configurations for the sections of the example filter.

Conf	Sec	LC-free ρ	used ρ	NG
1	1	(0.67,0.90)	0.875	1.58
	2	(0.56,0.97)	0.875	0.09
2	1	[0,0.46)	0.25	7.94
	2	[0,0.91)	0.875	0.22

5. CONCLUSIONS

The limit cycle problem in the delta DFIIt structure was addressed. It was shown that the complete absence of limit cycles cannot be guaranteed in the basic structure, although the amplitude of the limit cycles will be very low for narrowband lowpass filters. Furthermore, we found that with minor modifications to the delta operators in the structure the limit cycles can be completely eliminated when desired, however, at the cost of a slightly more complex implementation.

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