In Table IV, we give the Bhattacharyya lower bounds of order 1 (Cramer Rao lower bound) and of order 3 in the case of the logistic distribution. It should be noticed that the experimental results are close to the theoretical bounds.

3.3 Comparison with asymptotic formulas

In this section, We compare the output variance calculated by equation (6), with that given by the asymptotic approach of Huber.

The noise distributions belong to the family of generalized exponential noise distributions proposed by Bovik [BOV82]:

$$f_\lambda(t) = C \cdot \exp\left(-K|t|^\lambda\right).$$

This family contains the Exponential distributions ($\lambda = 1$), Gaussian distributions ($\lambda = 2$) and the Uniform distributions ($\lambda \to \infty$).

By using a result of Huber [HUB64], the asymptotic output variance of dα-filter for this family is given by:

$$Vh = \frac{1}{N} \cdot \frac{1}{\lambda^2 K^{2/\lambda}} \cdot \frac{\Gamma\left(\frac{2\alpha-1}{\lambda}\right) \cdot \Gamma\left(\frac{1}{\lambda}\right)}{\left[\Gamma\left(1 + \frac{\alpha-1}{\lambda}\right)\right]^2}$$

Γ is the ordinary gamma function.

Let Vapp be the output variance given by equation (6).

Let Vexp be the empirical output variance obtained with a sample of size 20000.

Fig3. displays the ratio Vexp/Vh ((a)-(b)) and Vexp/Vapp ((c)-(d)) as a function of the filter size N and for two noise distributions (uniform and exponential).

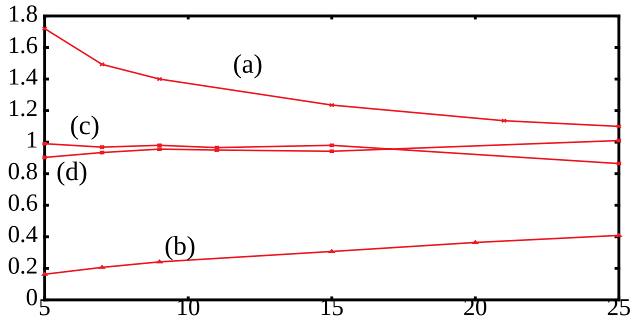

Fig 3. Ratio Vexp/Vh (uniform distribution (a) and exponential distribution (b)) and Vexp/Vapp (uniform distribution (c) and: exponential distribution (d)) versus N, α=4. Sample size 20000.

It can be seen that Vapp is a better estimate of the output variance than Vh for the lower values of filter size N.

4. CONCLUSION

We have investigated a new approach to the evaluation of performance of dα-filters. As we have seen, the main interest of using dα-filters is that they can be adapted to various noise distributions by tuning only one parameter (namely α). Moreover, optimal dα-filters are close to optimal L-filters. The performances of theses filters are very close to the theoretical lower bounds.

This approach can be extended to other M-filters defined by twice-differentiable cost function.

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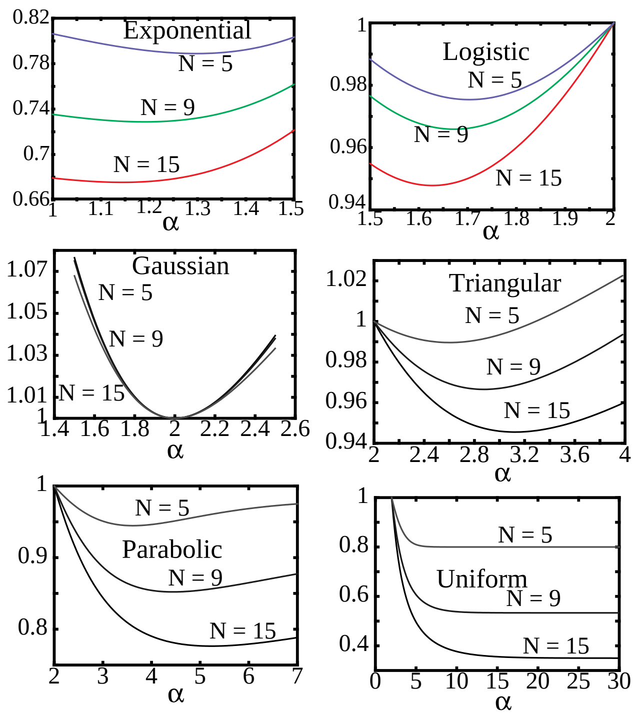

Fig 2. Output Variance versus α for different distributions (N = 5, 9, 15).

The optimal value of α depends greatly on the noise distribution [RIC64] as we can see in Fig2. Output variances are normalized with respect to that obtained with a mean filter (α=2).

TABLE I. Optimal α for different distributions and filter sizes.

Density/N	5	9	11	15
Uniform	∞	∞	∞	∞
Parabolic	3.61	4.45	4.75	5.25
Triangular	2.54	2.80	2.90	3.10
Gaussian	2.00	2.00	2.00	2.00
Logistic	1.70	1.65	1.64	1.62
Exponential	1.30	1.20	1.17	1.14

Table I lists the optimal value of α for each of the six noise distributions and for some odd values of N. It can be seen that for non gaussian distributions, the optimal values differ from α=2 (mean filter). Moreover, the gain in the output variance is noticeable.

In Table II, results show that optimal dα-filters are close to optimal L-filters as far as the output variance is concerned even for the Exponential and Logistic cases.

TABLE II. Comparison Between Theoretic Output Variance of Optimal L-filter and dα-filter for different distributions and filter sizes N. V: Optimal dα-filter Output Variance, VL: Optimal L-filter Output Variance.

N	V	VL
5	0.14	0.14
7	0.08	0.08
9	0.05	0.05
11	0.04	0.04
15	0.02	0.02

Uniform

N	V	VL
5	0.2	0.2
7	0.14	0.14
9	0.11	0.11
11	0.09	0.09
15	0.067	0.067

Gaussian

N	V	VL
5	0.196	0.192
7	0.14	0.136
9	0.108	0.104
11	0.088	0.084
15	0.064	0.060

Triangular

N	V	VL
5	0.18	0.18
7	0.12	0.12
9	0.09	0.09
11	0.07	0.07
15	0.05	0.05

Parabolic

N	V	VL
5	0.161	0.159
7	0.106	0.108
9	0.081	0.079
11	0.064	0.063
15	0.045	0.044

Exponential

N	V	VL
5	0.19	0.19
7	0.13	0.13
9	0.10	0.10
11	0.08	0.08
15	0.06	0.06

Logistic

3.2 Experimental results

The empirical output variances are estimated by applying dα-filters and optimal L-filters to stationary 512*512 size noisy images.

Results are presented in Table III. We can see that the output variance of the optimal L-filter and the optimal dα-filter are similar for each noise distribution.

TABLE III. Filter gain Vo/Vi - Comparison Between Optimal L-filter and Optimal Dα-filter.

	N = 9		N = 15	
Exponential	0.08	0.078	0.045	0.044
Logistic	0.105	0.105	0.062	0.061
Triangular	0.106	0.103	0.063	0.059
Parabolic	0.09	0.09	0.051	0.059
	L-filter	dα-filter	L-filter	dα-filter

TABLE IV. Efficiency of dα-filter and L-filter, Comparison with Bhattacharyya (order 3) and Cramer-Rao lower bounds, for unit variance logistic distribution.

Output Variance / N	5	7	9
Cramer-Rao	0.1824	0.1303	0.1013
Bhattacharyya (order 3)	0.1828	0.1304	0.1014
dα-filter (theoretic)	0.1923	0.136	0.1050
dα-filter (empirical)	0.1917	0.135	0.1053
optimal L-filter (theoretic)	0.191	0.1348	0.104
optimal L-filter (empirical)	0.1909	0.134	0.1041

with $\bar{\Phi} = \Phi\left(\bar{t}_1, \ldots, \bar{t}_N\right)$ and $\bar{t}_i = E\{t_i\}$ the expected value of t_i.
In the case of a symmetrical noise distribution, we have:

$$\sum\left(\left.\frac{\partial\Phi}{\partial t_i}\right|_{\left(\bar{t}_1, \ldots, \bar{t}_N\right)} \cdot \bar{t}_i\right) = 0 \ .$$

Without loss of generality we assume that $E\{x_i\} = 0$ since the dα-filter is a shift-invariant operator [BOL92a]. This implies $\bar{\Phi} = 0$.
Then,

$$\Phi(t_1, \ldots, t_N) = \sum\left(t_i \cdot \frac{\partial}{\partial t_i}\Phi\left(\bar{t}_1, \ldots, \bar{t}_N\right)\right) \qquad (3)$$

The filter output is then given by a linear combination of the order statistics of the input sequence: $y = \sum c_j \cdot t_j$.
For $\alpha \geq 2$, coefficients c_j are given by:

$$c_j = \frac{\partial\Phi\left(\bar{t}_1, \ldots, \bar{t}_N\right)}{\partial t_j} = \frac{\left|\bar{t}_j\right|^{\alpha-2}}{\sum\left|\bar{t}_i\right|^{\alpha-2}} \qquad (4)$$

We notice that the filter coefficients are symmetrical and verify the unbiasedness condition: $\sum c_i = 1$.

The case $\alpha < 2$ is troublesome because, for symmetrical distributions, one has $\overline{t_{n+1}} = 0$ [DAV81] which implies $c_j = 0$ for $j \neq n+1$ and $c_{n+1} = 1$. In fact, This is true only for $\alpha = 1$ (median filter).
For $1 < \alpha < 2$, the output is assumed to be close to the median value t_{n+1}. Let $y = t_{n+1} + \lambda$

The coefficients are then given by:

$$c_j = \frac{\left|\lambda - \left|\bar{t}_j\right|\right|^{\alpha-2}}{\sum\left|\lambda - \left|\bar{t}_i\right|\right|^{\alpha-2}} \qquad (5)$$

λ may be regarded as a typical deviation.
Let $\lambda = \delta \cdot \left(\overline{t_{n+2}} - \overline{t_{n+1}}\right)$.

By studying the influence of δ on the output variance, we experimentally found that choosing $\delta = 0.1$ yields to the best results (see Fig1.).

2.3 Output variance

The output variance is then computed as in [BOV83]:

$$\mathrm{Var}(\Phi) = c^t H c \qquad (6)$$

with H the Correlation Matrix of the order statistics and $c^t = (c_1, \ldots, c_N)$ is given by (4) or (5).

By plotting the output variance as a function of α, it is possible to find the optimal value of α (Fig2.).

The main interest of using dα-filter and the output approximating is that we have only one parameter (namely α) to tune instead of N for a L-filter (the filter coefficients).

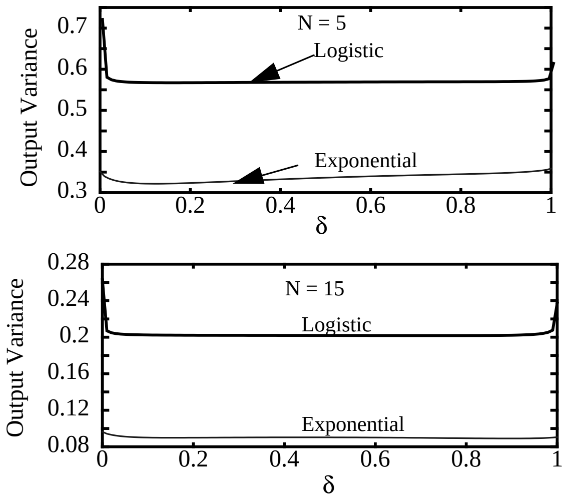

Fig1. Output Variance as function of δ for Exponential and Logistic distributions - Input Variance Vi = 1.

3. RESULTS

In this section, we study the output variance for different noise distributions.

In the following, Vo is the Output Variance and Vi is the Input Variance.

3. 1 Theoretical results

We give some theoretical results for different noise distributions: Uniform, Gaussian, Triangular, Parabolic, Logistic and Exponential:

Uniform: $f(x) = \begin{cases} (1/(2\sqrt{3}\sigma)) ; |x| < \sqrt{3}\sigma \\ 0 \quad \text{otherwise} \end{cases}$

Gaussian: $f(x) = e^{-x^2/\left(2\sigma^2\right)}/(\sqrt{2\pi}\sigma)$

Parabolic: $f(x) = \begin{cases} 3(\sqrt{5}\sigma + x)(\sqrt{5}\sigma - x)/\left(20\sqrt{5}\sigma^3\right) ; |x| < \sqrt{5}\sigma \\ 0 \quad \text{otherwise} \end{cases}$

Exponential: $f(x) = e^{-|x|/\sigma}/(2\sigma)$

Triangular: $f(x) = \begin{cases} 2(\sqrt{6}\sigma/2 - |x|)/(3\sigma) & |x| < \sqrt{6}\sigma/2 \\ 0 & \text{otherwise} \end{cases}$

Logistic: $f(x) = \dfrac{e^{x/(\sigma\sqrt{3}/\pi)}}{(\sigma\sqrt{3}/\pi)\left(1 + e^{x/(\sigma\sqrt{3}/\pi)}\right)^2}$

where σ is the standard deviation.
These six distributions give a wide range of behaviour ranging from the fairly heavy-tailed exponential to the very shallow-tailed uniform densities.

Performance Evaluation of Dα-filters

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ABSTRACT

We study the output variance of a class of nonlinear filters, called dα–filters. In general, it is impossible to obtain an explicit expression of the output variance because of the implicit Input/Output relationship, except for α=1 (median filter), α=2 (mean filter) and $\alpha=\infty$ (midrange filter). In this paper, we develop a new approach to the computation of the filter output variance. It is based on a linearisation of the filter output about the order statistics expected values. This approximation is valid for $\alpha > 1$. It allows optimal α-values to be computed. Experimental results are presented. They are compared to those of L-filters and with theoretical lower bounds (Bhattacharyya system of lower bounds).

Keywords: Image processing, optimal filtering, order statistics, M-filtering, theoretical bounds, noise distributions.

1. INTRODUCTION

Dα-filter is a special case of M-filtering [PIT90]. Its output can be regarded as a nonlinear function of order statistics which depends on two parameters: N (filter size) and α (norm) [BOL92a]. The main objective of such filters are noise reduction and preservation of the structures viewed in the image to improve the efficiency of the segmentation step.
In this paper, we use an approximation of the output filter which allows us to express the output variance with an analytical expression about α and N. Since the output variance is now a continuous function about α, we can study the performances of dα-filtering by plotting the output variance and then selecting the optimal value. Moreover, we compare the performances of this filter with those obtained by the optimal L-filter [BOV82]. The results show that dα-filters are close to optimal L-filters as far as the output variance is concerned.
We have also studied the efficiency of dα-filtering by using the Bhattacharyya system of lower bounds [KEN76]. The experimental results are close to the theoretical bounds.

2. Dα-FILTER

2.1 Model and Criterion

We consider a dα-filter of size N operating on a sequence $\{x_1,...,x_N\}$, for N = 2n+1. Arrange the x_i in ascending order of magnitude and denote set by $\{t_1,..., t_N\}$, so that $t_1 \le t_2 \le \ldots \le t_N$. t_r is the rth order statistic. We suppose the x_i result from a constant signal corrupted by zero-mean white noise: $x_i = s + u_i$.
The u_i are iid random variables such that $E\{u_i\} = 0$. In addition, it will be assumed that the noise distribution is symmetrical.
The filter output y is defined by means of a cost function χ by: $y = \text{Argmin}\, \varepsilon(u)$

with $\varepsilon(u) = \sum_i \chi\left(u - t_i\right)$.

Function χ is chosen here as: $\chi(\tau) = |\tau|^{\alpha}$, $\alpha \ge 1$. This filter differs from the γ-filter [AST89]. In the latter case, the output is constrained to belong to the input data set.
The convex function ε has a unique minimum such that:

$$\frac{\partial\varepsilon}{\partial y} = \varphi\left(y, t_1, .., t_N\right) = \alpha \cdot \sum \text{sgn}\left(y - t_j\right) \cdot \left|y - t_j\right|^{\alpha - 1} = 0$$

This equation defines an implicit function Φ of N variables: $y = \Phi(t_1,..., t_N)$.
Though function Φ is not explicit, we can express its partial derivatives with respect to the order statistics:

$$\frac{\partial\Phi}{\partial t_j} = -\frac{\frac{\partial\varphi}{\partial t_j}}{\frac{\partial\varphi}{\partial y}} = \frac{\left|y - t_j\right|^{\alpha - 2}}{\sum \left|y - t_i\right|^{\alpha - 2}} \qquad (1)$$

It has been shown [BOL92b] that $\partial\Phi/\partial t_j$ is continuous at t_j=y, since $\partial\Phi/\partial t_j$ tends towards 1 as t_j tends towards y.
Furthermore, The partial derivatives $\partial\Phi/\partial t_j$ are non-zero only for $t_j \ne y$.

Then, we can expand $\Phi(t_1,..., t_N)$ which is at least C1 in Taylor series about order the statistics expected values $E\{t_i\}$.

2.2 Output approximation

The linearisation of Φ about the order statistics expected values leads to:

$$\Phi(t_1, \ldots, t_N) = \bar{\Phi} + \sum\left(\left(t_i - \bar{t}_i\right) \cdot \left.\frac{\partial\Phi}{\partial t_i}\right|_{\left(\bar{t}_1, \ldots, \bar{t}_N\right)}\right) \qquad (2)$$