MULTIRESOLUTION ANALYSIS USING ORTHOGONAL POLYNOMIAL APPROXIMATION

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Abstract

Multiresolution decomposition of signals has been conventionally carried out by the wavelet representation. In this paper, the orthogonal polynomial approximation has been employed for multiresolution analysis. It is demonstrated that the proposed technique based on polynomial approximation has certain distinct advantages over the conventional method employing wavelet representation.

1 Introduction

Multiresolution representations have been proven to be very effective for analyzing the information content of signals and images [1–6]. Multiresolution analysis refers to approximating the signal at various resolutions, computing the details (the differences of information between the approximations at current and next resolutions) of the signal, and then interpreting the signal information as depicted in approximations and details. Conventionally, the multiresolution representations of a signal are obtained by decomposing the signal in a wavelet orthonormal basis. The wavelet decomposition can be interpreted as a signal decomposition in a set of independent frequency channels with orientation in time or space. The wavelet orthonormal bases are generated on translation and dilation of the special functions called wavelets. It is important that the elements of orthonormal bases have good localization properties in both the temporal (or spatial) and Fourier domains [5].

Considering the great potentiality of multiresolution signal decomposition in applications like pattern recognition, it is imperative that the study of operators which approximate a signal at various resolutions will be highly desirable. In this paper, we propose an alternative multiresolution representation of a signal based on the orthogonal polynomial approximation [7]. The analysis technique presented here is conceptually simpler and easier to implement than the wavelet representation.

2 Orthogonal Polynomial Approximation

Let $f(x)$ be a function and $\{x_i; i = 0, ..., n-1\}$ be a sequence of sampling points, not necessarily uniformly spaced, at which the observed value of the function is $\bar{f}_i$ and $f_i = f(x_i)$ is the true value. The orthogonal polynomial appроimation of order m for the function is given by

$$f^m(x) = \sum_{j=0}^{m} c_j p_j(x) \qquad (1)$$

where the set of polynomials $\{p_j(x)\}$ of degree j is orthogonal over the sampled points,

$$\sum_{i=0}^{n-1} p_l(x_i) p_k(x_i) = 0 \qquad \text{for} l \neq k \qquad (2)$$

and, there exists a recurrence relation for generating the polynomials:

$$p_{j+1}(x) = (x - a_{j+1}) p_j(x) - b_j p_{j-1}(x) \qquad j \geq 0 \quad (3)$$

with $p_0(x) = 1, p_{-1}(x) = 0$, and

$$a_{j+1} = \frac{\sum_{i=0}^{n-1} x_i [p_j(x_i)]^2}{\sum_{i=0}^{n-1} [p_j(x_i)]^2},$$

$$b_j = \frac{\sum_{i=0}^{n-1} [p_j(x_i)]^2}{\sum_{i=0}^{n-1} [p_{j-1}(x_i)]^2}.$$

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The coefficients of the polynomial series are computed in the least squares sense as

$$c_j = \frac{\sum_{i=0}^{n-1} \bar{f}_i p_j(x_i)}{\sum_{i=0}^{n-1} [p_j(x_i)]^2}. \tag{4}$$

Moreover, when the order of approximation, m is chosen by the criterion of minimum error–variance, the uncorrelated part (noise) of the signal can be rejected [7].

3 Multiresolution Analysis

Let P_j be the operator which approximates a signal with a combination of first j orthogonal polynomials. Then it is easy to verify that P_j is a linear projection operator on a vector space W_j spanned by these polynomials, and among all the approximated functions in W_j, $P_j f(x)$ is the function which is most similar to $f(x)$ (least squares approximation) [8]. Moreover, the approximation of a signal by first j polynomials contains all the necessary information to compute the same signal with first k polynomials for all $k \leq j$. Furthermore, the approximation $P_j f(x)$ of a signal $f(x)$ can be characterized by $j+1$ samples (interpolation formula) [8], and when $f(x)$ is translated by some length, $P_j f(x)$ is translated by the same amount. It is also to be noted that since we have n samples of the signal, the approximated signal $P_j f(x)$ will converge to the original signal as j increases to $n-1$. Conversely, as j decreases to 0, the approximated signal settles to a constant value.

The above discussion shows that except for the property by which the spaces of approximated functions should be derived from one another by scaling (similarity property) [5], all the properties of multiresolution approximation hold for the polynomial approximation. Since the property of similarity does not hold for the polynomials, the different basis functions can not be obtained by dilating one function. Instead, the different basis functions in this case are obtained by the recurrence relation (3).

4 Simulation Study

In the simulation study, a set of 190 samples of the electrocardiogram (ECG) signal is processed. The plot of the signal is shown in Fig. 1. The discrete time when multiplied by the sampling interval (1/250 s) provides the continuous time. The multiresolution decomposition of the signal is obtained in turn by the wavelet representation and polynomial approximation. For the wavelet transform, we implement the algorithm given by Mallat [4] using the functions developed by Cody [9]. We have used the cubic spline wavelets ($\alpha = -1.0, \beta = 0.5$). For the polynomial approximation, the optimum degree of polynomial is found to be 45. Figs. 2 to 10 show all the graphs for comparison of corresponding signals at various resolutions as obtained by the wavelet representation and polynomial approximation with appropriate degree. At the optimum (degree 45) polynomial approximation, the approximated signal is close to the signal at resolution 1/4 indicating that the original signal has been corrupted with noise.

Figs. 11 to 18 show the power spectral density (PSD) plots for the signals at different resolutions. As we can observe from the PSD plots of the corresponding approximated signals, the peak patterns are slightly different but the spreads of the spectra are same in all the cases. Clearly then, the simulation study shows that the orthogonal polynomial approximation can be used for multiresolution analysis. The frequency contents of polynomials of different degrees are shown in Figs. 19 and 20. It can be seen from the graphs that these polynomials occupy different frequency bands with slight overlaps.

5 Discussions and Conclusions

We know that the wavelets have the varying time localization property [6]. Indeed, the polynomials also have the varying time localization property. Suppose we have a data sequence which has n samples, and we wish to approximate it with an $n-1$ degree polynomial. If we change one of the samples in the original data, only that sample in the fitted curve will change because for this degree, the polynomial reproduces all the points exactly. This is the case of narrowest time window. Now as we decrease the degree of the polynomial, the curve becomes smoother and the change in one sample of the input starts propagating through neighboring samples. Eventually, when we approximate the signal with a zero degree polynomial, the output is the d.c. value of the signal and change in one sample of the input results in a uniform change in all the samples of the output. This is the case of the widest time window.

Finally, the orthogonal polynomial approximation has certain advantages over the wavelet representation as listed below:

1. In the case of wavelet decomposition, the step

size is fixed, whereas with the polynomial approximation one can choose much smaller step size for more detailed information.

2. With polynomial approximation, the decomposition and reconstruction of a signal are the one–step processes with direct implementation.

3. The polynomial approximation has inherent noise rejection capability (using minimum error–variance criterion) [7]. This allows us to identify the noise part of a signal, which generally has no significance in multiresolution analysis.

4. Since the orthogonal polynomials are generated with arbitrarily spaced points, the approximation can be used for nonuniformly sampled data [7].

References

[1] A. Grossmann and J. Morlet, "Decomposition of Functions into Wavelets of Constant Shape and Related Transforms", in *Mathematics and Physics, Lectures on Recent Results*, Singapore: World Scientific, 1985.

[2] R. Kronland-Martinet, J. Morlet and A. Grossmann, *Analysis of Sound Patterns through Wavelet Transform*, preprint CTP-**87**/p. 1981, Centre de Physique Théorique, CNRS, Marseille, France, 1987.

[3] J. Daubechies, "Orthonormal Basis of Compactly Supported Wavelets", *Communications on Pure and Applied Math.*, **41**(7), pp. 909–996, Oct. 1988.

[4] S.G. Mallat, *A Theory for Multiresolution Signal Decomposition*, Ph.D. thesis, Univ. of Pennsylvania, Dept. of Elect. Eng. and Comput. Sci., 1988.

[5] S.G. Mallat, "A Theory for Multiresolution Signal Decomposition: The Wavelet Representation", *IEEE Trans. on Pattern Anal. and Machine Intelli.*, **11**(7), pp. 674–693, Jul. 1989.

[6] J. Daubechies, "The Wavelet Transform, Time Frequency Localization and Signal Analysis", *IEEE Trans. on Inform. Theory*, **36**(5), pp. 961–1005, Sep. 1990.

[7] P. Sircar and A.C. Ranade, "Nonuniform Sampling and Study of Transient System Response", *Proc. IEE, Part F*, **139**(1), pp. 49–55, Feb. 1992.

[8] A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, Singapore: McGraw-Hill, 1988.

[9] M.A. Cody, "The Fast Wavelet Transform", *Dr. Dobb's Journal*, pp. 16–28, Apr. 1992.

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