Order Determination of State Space Systems

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ABSTRACT

Recent techniques proposed for the identification of state space models have focused on using the singular value decomposition of block Hankel input-output matrices. In these procedures the order of the system is determined by examining the singular values and identifying the separation between the "signal" and "noise" subspaces. Order determination of state space systems requires an understanding of what singular value magnitudes are expected. This paper examines how system structure and noise levels affect the magnitude of singular values. An order selection criterion formed from the AIC and MDL is also examined.

1 Introduction

Model order determination aims to find the structural order of a system, given input and output observations of the system. For state space models of systems, model order determination depends on the correct interpretation of the singular values calculated from a specially constructed block Hankel matrix [1]. The relationship between the system structure and the resulting singular values and singular vectors must be understood, so that the correct interpretation can be made. The effect of additive white noise is known to increase the "noise" singular values to $\sqrt{j}\sigma_N^2$, where j is number of columns of the Hankel matrix [2, 3]. In this study we are examining the changes in order-determining singular values due to system structure so that a suitable noise level criterion can be established to allow reliable order determination.

Model order determination of systems in which only the output is observed include Akaike's information theoretic criterion (AIC) [4] and Rissanen's minimum description length (MDL) [5] which are used to identify the optimal polynomial based model [6, 7]. These ideas have also been developed for the state space identification of stochastic models and signal detection algorithms [8, 9, 10]. The difference between these methods and state space identification of deterministic models is that the Hankel matrix for stochastic systems and signal detection only contains one measured variable and the singular values can be directly related to "signal" and "noise" power. In the deterministic model case, the Hankel matrix contains input-output information and cannot be used in the same way [11].

1.1 Subspace Spectral Estimation

Subspace spectral estimation techniques, such as MUSIC, are based on obtaining eigenvalues and eigenvectors of the data correlation matrix that are associated with the noise subspace [10]. This can also be achieved by using the singular values and singular vectors of a Hankel matrix constructed from the data. Determining the noise subspace requires the examination of the eigenvalues (or singular values) magnitudes and separating the "signal" and "noise" subspaces [2].

Wax and Kailath developed the Akaike Information Criterion (AIC) and the Minimum Description Length (MDL) that can be applied to resolving this problem [8].

The AIC and MDL are defined in terms of the likelihood function as:

$$\mathrm{AIC}(k) \;=\; -2\ln f_{x;\theta}(x;\theta) \;+\; 2k \tag{1}$$

$$\mathrm{MDL}(k) \;=\; -\ln f_{x;\theta}(x;\theta) \;+\; \frac{1}{2}k\ln K, \tag{2}$$

where k is the number of free parameters and K is the number of vectors used to estimate the parameters of the density distribution function $f_{x;\theta}(x;\theta)$.

Wax and Kailath showed that these criteria can be expressed explicitly in terms of the eigenvalues, λ, the dimension of the correlation matrix, M, and the data sequence length, K, [8, 10]:

$$AIC(M) = -2K(N-M)\ln\varrho(M) + 2M(2N-M) \tag{3}$$

$$MDL(M) = -K(N-M)\ln\varrho(M) + \frac{1}{2}M(2N-M)\ln K \tag{4}$$

where

$$\varrho(M) = \frac{(\lambda_{M+1}\lambda_{M+2}\ldots\lambda_N)^{\frac{1}{N-M}}}{\frac{1}{N-M}(\lambda_{M+1}+\lambda_{M+2}+\ldots+\lambda_N)} \tag{5}$$

The requirements associated with the order determination of the signal space parallels the system order determination problem in state space system identification.

1.2 State Space System Identification

State space system identification for both stochastic and deterministic systems requires the identification of the system order or equivalently, the dimension of the state sequence. Methods have been proposed to provide indications of the system order which requires estimating the dimension of a subspace projection or the angles between two subspaces [11, 12].

The input block Hankel matrix U is defined as

$$U = \begin{bmatrix} u_0 & u_1 & \dots & u_{j-1} \\ u_1 & u_2 & & u_j \\ \vdots & & \ddots & \\ u_{2i-1} & \dots & & u_{j+2i-2} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{7}$$

The output block Hankel matrix, Y, is similarly defined.

Order determination of a stochastic system is identified as the dimension of a subspace, $\mathsf{Range}(\mathcal{P})$, formed by projecting the row space of Y_2 (the "future" output) onto the row space of the Y_1 (the "past" outputs).

$$\mathcal{P} = (Y_2 Y_1^T)(Y_1 Y_1)^{-1} Y_1 \tag{8}$$

It can be shown that for large number of output samples, j, the column space of this matrix spans the column space of the observability matrix [11]. Similarly the column space of the matrix generated from projecting the row space of Y_1 onto the row space of Y_2 spans the column space of the controllability matrix. Hence as j approaches ∞, the rank of $\mathcal{P}$ will equal the system order, n [11]. The angles between these two subspaces $\mathsf{Range}(Y_1^T)$ and $\mathsf{Range}(Y_2^T)$ can also be used to determine the system order, since as j approaches ∞ there will be n angles different from $\pi/2$.

The method for determining the system order of a deterministic state space system is proposed in the N4SID algorithm [1]. In this algorithm the order is identified as the dimension of a subspace formed from the intersection of $\mathsf{Range}(H_2^T)$ and $\mathsf{Range}(H_1^T)$ [11, 1]

$$\mathsf{dim}\left(\mathsf{Range}(H_1^T) \cap \mathsf{Range}(H_2^T)\right) = n, \tag{9}$$

where H_1 and H_2 are defined as

$$H_1 = \begin{bmatrix} U_1 \\ Y_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} U_2 \\ Y_2 \end{bmatrix}. \tag{10}$$

The columns of an $n \times j$ matrix, whose rows are a set of independent basis vectors of the intersection space, form a valid state sequence.

Alternatively the order of a deterministic system can be identified directly from the rank of

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \tag{11}$$

This method however does not aid in the generation of a valid state sequence. In the absence of noise the rank of H is equal to $2i+n$. If the rank is determined using the SVD, the effect of white uncorrelated noise on the singular values has been clearly defined [2, 13]. What is not known is the magnitude of the singular values associated with the signal space. Any that are less than the noise singular values will not be identified.

2 Order Determination

To identify the system order requires being able to determine the effect that system structure has on the system identification procedures and to be able to correctly interpret the results produced from the algorithms.

This paper aims at addressing these issues through

I examining the effect of pole location on the order determination process;

II examining the development of the AIC and MDL criterion for state space systems.

2.1 Order determination dependence on system structure

We know that the singular values of the matrix H in (11), used in determining the system order, change with the system pole-zero locations and noise power. When the system pole and zero locations are changed, the correlation and cross correlation between input and output, and consequently the angles between the row vectors of H, change. We wish to quantify the size of the changes corresponding to the changes to pole and zero locations.

The order of a deterministic system can be identified by examining the number of significant (non-zero) singular values of the input-output block Hankel matrix, H [2], where

$$H = \begin{bmatrix} U_1 \\ Y_1 \\ U_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \dots & & \sigma_{4i} \end{bmatrix} V^t. \tag{12}$$

We have performed an experiment using a a second order system with a complex conjugate pole pair positioned at various locations inside the unit circle in the $\mathcal{Z}$-plane. Noise was added to the system. The input and output signal power is maintained equal and constant for all simulations. Figure 1 shows the effect of pole location and signal power on the order determining singular value σ_{2i+n}, and the first noise singular value, σ_{2i+n+1}. These are plotted against the system pole location (with positive imaginary component), forming two planes. For poles located near $z = \pm 1$ or $z = 0$, the separation between the order determining and noise singular values reduces significantly. In this example, having SNR $20dB$, systems with poles inside the circle, centered on

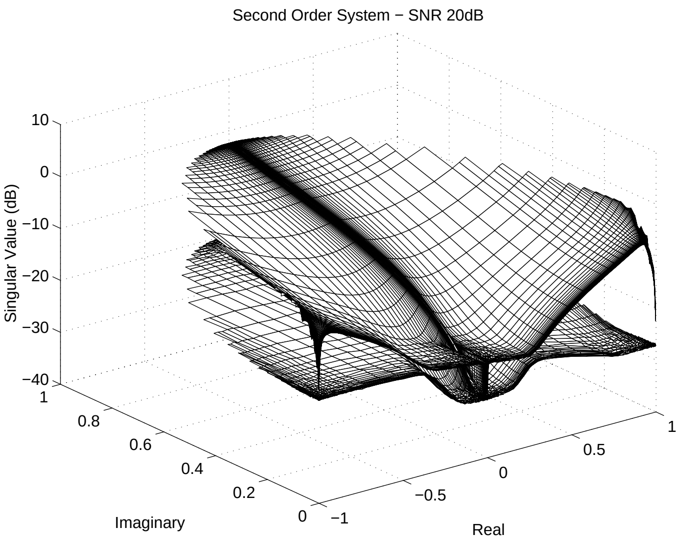

Figure 1: Order determining and first "noise" singular values plotted against the complex conjugate pole pair location of a second order system

the origin with radius approximately 0.3, have the order determining singular values which can not be reliably distinguished from the noise singular values.

A second experiment was performed using real, repeated poles. Figure 2 shows the magnitude of the order determining singular value for systems with various pole locations along the real axis and various pole multiplicity, with the input and output signal power set equal and constant. This figure shows a significant increase in the required SNR for the order determining singular value to remain above the noise floor, as the pole multiplicity increases.

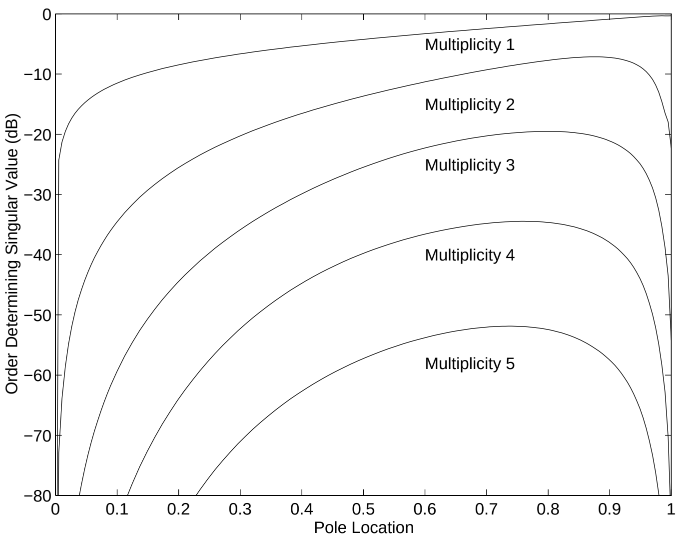

Figure 2: Order determining singular values for a systems with real repeated poles plotted against the location of a pole

Figure 2 shows that simple (non-repeated poles) can be identified for moderate SNR. The diminishing size of order-determining singular values for poles near $z = 0$ is due to the decreased correlation between the current outputs and previous outputs. The output row vectors align themselves with the input row vectors, reducing the power in the direction orthogonal to the input. For poles near $z = 1$ the output row vectors are highly correlated so that the signal power concentrates in one direction. This does not occur in the simple pole case since the increased correlation in the output and persistent excitation of the input maintains the $\mathsf{dim}(\mathsf{span}(H)) = 2i + 1$.

In a third experiment, the algorithm for N4SID [1] was used with the same conditions as in the first experiment. The N4SID method does not produce any better results than the earlier experiment. This method finds the intersection between 2 row spaces. The intersection is reduced as the two row spaces involves contract due to increased correlation between the rows.

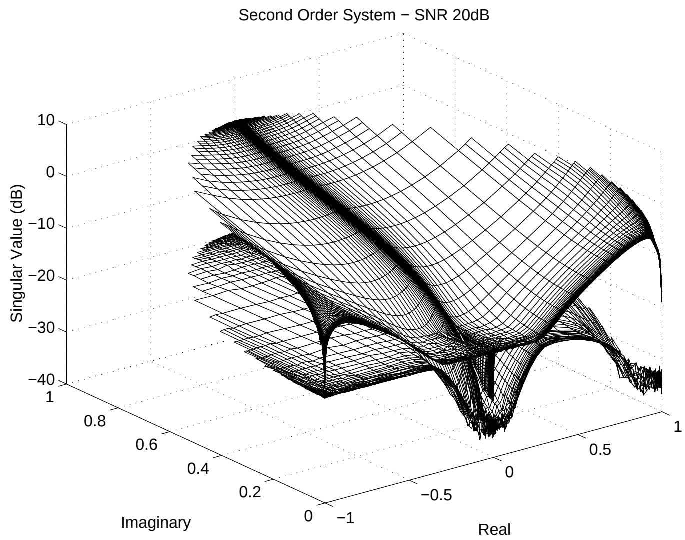

Figure 3: Order determining and first "noise" singular values plotted against the complex conjugate pole pair location of a second order system

2.2 System Order Selection

The order can be chosen by visual inspection of a logarithmic plot of the singular values, assuming that the flat region of the curve represents noise. For minimal supervision, the AIC or MDL techniques can be used.

The order determination process described for the spectral estimate techniques parallel the requirements for state space system identification. A modified form of equations (2) - (5) could be used after taking into account the different degrees of freedom associated with a real matrix.

Alternatively the AIC and MDL can be written as [6]:

$$AIC = \mathsf{min}_\theta \left\{ -2 \left[\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 \right] + 2\,\mathsf{dim}(\theta) \right\} \quad (13)$$

$$MDL = \mathsf{min}_\theta \left\{ - \left[\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 \right] + \frac{1}{2}\mathsf{dim}(\theta) \cdot \mathsf{log}(N) \right\} \quad (14)$$

In the second term, $\dim(\theta)$ is equal to the number of independent model parameters. The first term can be calculated from the residual error, ε, of the ML estimator, for a particular model choice.

3 Conclusion

The results show a relationship between the structure of the system being identified and the singular values used in the order determination process. This has been extended to show that the system structure also defines a minimal signal-to-noise ratio for reliable order determination.

An AIC and MDL measure for state space deterministic models was also examined. From this information it is possible to begin to understand some of the limitations imposed by a system structure which should be considered when attempting to identify a system.

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