ROBUST PARAMETER ESTIMATION FOR PERIODIC POINT PROCESS SIGNALS USING CIRCULAR STATISTICS

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ABSTRACT

We discuss the application of signal parameter estimators for periodic point process signals with missing data. The proposed estimation techniques operate on the observed event arrival time sequence of a pulse train signal and have application to pulse train signal classification and signal reconstruction. The methods we describe are based on the use of circular statistics and are shown to offer considerable robustness to a pulse train time series corrupted by missing pulses.

1. INTRODUCTION

The estimation of parameters associated with *point process* signals is an important signal processing problem in a number of fields (e.g., see [2]). In the interception of pulsed radar signals through the deployment of a passive radar intercept sensor, for instance, one is often interested in obtaining accurate estimates of the pulse train periodicities and related signal parameters for electronic defence purposes (e.g., see [3–5, 8]). The point process model of the pulse train signal considered here is useful for developing the proposed estimation techniques which operate on the event *time of arrival (TOA)* sequence of the recorded pulse stream. In this paper we will examine estimation techniques for a point process signal with *known* periodicity. In practice, the periodicity would be estimated using one of a number of available methods (e.g., see [4, 8]). Specifically, we describe robust techniques for the estimation of the pulse train *TOA phase* and *jitter variance* for a data sample in which a significant number of pulses are missing. The problem of missing observations occurs frequently in the case of signals captured by a radar intercept receiver and which motivates our work. The methods we describe have application to signal classification and signal reconstruction. For example, a number of pulse train components associated with a single signal but fragmented through pre-processing may be re-associated by comparison of each component's TOA jitter variance. Characterising the amount of jitter on a pulse train signal may also assist with the identification of a radar source.

In the case of a pulse train without missing observations, standard maximum likelihood and least squares methods can be applied to obtain estimates of a pulse train's period, TOA phase and jitter variance [5]. However, for incomplete data records the canonical techniques become unreliable and one must adopt alternative estimation schemes. In this paper, we take a novel approach by examining the use of estimation techniques for *circular* data. We show how the methods may be applied to an isolated pulse train through a simple transformation of the observed event arrival time sequence. Moreover, we will demonstrate that the circular statistics based estimators offer considerable robustness to missing pulses. The methods are also shown to be statistically consistent and asymptotically unbiased and efficient.

2. POINT PROCESS SIGNAL MODEL

Since we will process pulse train data solely on the basis of the observed pulse or event arrival time sequence, an appropriate time series model for the signal of interest is one based on a series of delta functions, yielding a point process signal. A pulse train signal, $p(t)$ say, may then be modelled by

$$p(t) = \sum_{k=0}^{K-1} \psi_k \delta(t - t_k), \tag{1}$$

where the TOA of the kth pulse is denoted by t_k and is likely to be measured by a passive intercept receiver using leading edge timing circuitry. The $\{t_k\}$ form a time ordered sequence with $t_{k+1} > t_k$, $\forall k$. It will be assumed that N out of a possible K pulses are recorded over a finite observation interval $[0,\, T_{obs}]$, and the pattern of missed pulses is determined by the indicator function, ψ_k, where

$$\psi_k = \begin{cases} 1, & \text{pulse observed,} \\ 0, & \text{pulse missing.} \end{cases} \tag{2}$$

Furthermore, the sequence $\{\psi_0,\ \psi_1,\ \ldots,\ \psi_{K-1}\}$ forms a set of independent and identically distributed (i.i.d.) binary random variables with probabilities

$$\Pr(\psi_k = 1) = p \qquad \text{and} \qquad \Pr(\psi_k = 0) = 1 - p = q. \tag{3}$$

We also assume that the TOA sequence of events is nominally periodic but deviates from a perfectly periodic sequence due to random timing variations termed *jitter*. With a slight abuse of our previous notation for $\{t_k\}$, the model for the nth pulse TOA with a non-cumulative timing perturbation is

$$t_n = t_\phi + k_n T + \epsilon_n, \qquad n = 0, 1, \ldots, N-1, \tag{4}$$

where T denotes the pulse train periodicity, t_ϕ the TOA phase which defines the start of the pulse train arrival time sequence relative to an arbitrary time origin such that $t_\phi \in [0,\, T)$, and ϵ_n a non-cumulative jitter component. The $\{k_n\}$ correspond to the set $k_n \in \{0, 1, \ldots, K-1\}$ of unknown integers and allow for the possibility of missing data. For the case of no missing data, $k_n = n$, $\forall n$. The $\{\epsilon_n\}$ are taken to be i.i.d. random variables with a truncated Gaussian distribution $\mathcal{N}(0,\, \sigma_\epsilon^2)$, such that $|\epsilon_n| \leq 0.5T$. Finally, we will use the term *percentage jitter* to characterise the magnitude of the TOA perturbations and define percentage jitter to be $300\sigma_\epsilon/T$, which corresponds to the three sigma level of a Gaussian jitter distribution.

3. ESTIMATION TECHNIQUES FOR CIRCULAR DATA

The use of circular statistics is appropriate for data which can be viewed as observations on the *unit circle* as opposed to data that represent observations on the *real line* and are discussed in detail in [7]. In some cases one might wish to move from one domain to the other. The time series data considered here exhibit a cyclic behaviour of known periodicity and this allows one to readily transform the observation sequence of event arrival times from the linear domain into *phase* measurements on the unit circle that have period 2π.

3.1. TOA Phase and Jitter Variance Estimators

Given a sequence of N pulse arrival times $\{t_n\}$, each event TOA may be transformed into a circular observation through a *folding* or *wrapping* operation. Specifically, the phase $\phi(t_n)_{2\pi}$ associated with the nth arrival time is computed from

$$\phi(t_n)_{2\pi} = 2\pi\left(\frac{t_n}{T} - \left\lfloor\frac{t_n}{T}\right\rfloor\right), \qquad n = 0, 1, \ldots, N-1, \tag{5}$$

where $\lfloor\cdot\rfloor$ denotes the largest integer smaller than or equal to its operand and the subscript 2π on the sequence $\{\phi(t_n)_{2\pi}\}$ indicates that the phases are evaluated modulo 2π. The TOA sequence, $\{t_n\}$, can now be regarded as being wrapped around a circle of circumference equal to the folding period T. Alternatively, the phase sequence $\{\phi(t_n)_{2\pi}\}$ is mapped onto the unit circle with circumference 2π. The phase distribution obtained from the folding operation will be determined by the nature of the pulse arrival time deviations about a strictly periodic sequence with period T. For Gaussian distributed timing variations, TOA folding yields the wrapped normal phase distribution, $f(\phi)$, where [7]

$$f(\phi) = \frac{1}{\sigma_\phi\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}\exp\left[-\frac{(\phi-\mu_\phi+2\pi k)^2}{2\sigma_\phi^2}\right], \quad 0 \le \phi < 2\pi, \tag{6}$$

and the circular mean and variance of the distribution are specified by μ_ϕ and σ_ϕ^2 respectively. One may readily see that folding is robust to missing pulses in that the character of the underlying parent phase distribution (6) is preserved, e.g., by constructing a phase histogram of folded TOAs with and without missing data. This robustness motivates the use of circular estimation methods as a basis for estimating parameters associated with pulse train TOA measurements.

Given a phase data vector $\boldsymbol{\phi} = [\phi_0\ \phi_1\ \cdots\ \phi_{N-1}]^T$, where for ease of notation we take $\phi_n \equiv \phi(t_n)_{2\pi}$, and where $\boldsymbol{\phi}$ is associated with a unimodal phase distribution, estimates of the circular mean, μ_ϕ, and variance, σ_ϕ^2, of the parent phase distribution can be obtained from [7]

$$\hat{\mu}_\phi = \arg\left[\sum_{n=0}^{N-1}\exp(i\phi_n)\right], \qquad 0 \le \hat{\mu}_\phi < 2\pi, \tag{7}$$

and

$$\begin{aligned}\widehat{\sigma_\phi^2} &= -2\ln(1-\hat{V}_\phi), \qquad 0 \le \widehat{\sigma_\phi^2} < \infty,\\ &= -2\ln r,\end{aligned} \tag{8}$$

where

$$\hat{V}_\phi = 1 - r, \qquad 0 \le \hat{V}_\phi \le 1, \tag{9}$$

and

$$r = \frac{1}{N}\left|\sum_{n=0}^{N-1}\exp(i\phi_n)\right|, \qquad 0 \le r \le 1. \tag{10}$$

Note that $\hat{V}_\phi$ is also referred to as an estimate of circular variance, but is restricted to the range $0 \le \hat{V}_\phi \le 1$, so that one cannot directly compare $\hat{V}_\phi$ to the variance on the real line since the latter takes values in $[0, \infty)$ [7]. For the event TOA model of Equation (4), the wrapping operation using the period T yields for the nth pulse arrival time

$$\phi(t_n)_{2\pi} = \frac{2\pi t_\phi}{T} + \frac{2\pi\epsilon_n}{T}, \tag{11}$$

and the phases $\{\phi(t_n)_{2\pi}\}$ will be distributed according to the wrapped normal distribution with circular mean $\mu_\phi = 2\pi t_\phi/T$ and circular variance $\sigma_\phi^2 = 4\pi^2\sigma_\epsilon^2/T^2$. Hence, given an event TOA data vector $\boldsymbol{t} = [t_0\ t_1\ \ldots\ t_{N-1}]^T$ (potentially with missing observations) and the block circular estimators of Equations (7) and (8), one may obtain estimates of a pulse train's TOA phase, t_ϕ, and jitter variance, σ_ϵ^2, through the relations

$$\hat{t}_\phi = \frac{T}{2\pi}\hat{\mu}_\phi, \tag{12}$$

and

$$\widehat{\sigma_\epsilon^2} = \left(\frac{T}{2\pi}\right)^2\widehat{\sigma_\phi^2}. \tag{13}$$

It is important to note that the application of linear estimators to the phases $\{\phi(t_n)_{2\pi}\}$, as opposed to circular estimators, would yield erroneous results if the phase distribution straddled the $0 - 2\pi$ boundary.

3.2. Comparison with Other Work

We note that the estimate of t_ϕ provided by Equation (12) is identical to that suggested in [4] using the discrete Fourier transform (DFT), $X(\omega)$, of the delta function time series model of Equation (1). Furthermore, the periodogram estimate, $\hat{P}(\omega)$, of power spectral density based on $X(\omega)$ for the same time series model comprising N events is related to the mean vector length r, introduced above in the context of circular statistics, viz

$$\hat{P}(\omega) = \frac{|X(\omega)|^2}{N} = Nr^2, \tag{14}$$

where $X(\omega) = \sum_{n=0}^{N-1}\exp(-i\omega t_n)$ and ω denotes angular frequency. A periodogram spectral estimate of this type is known as a *spectrum of counts* in the point process literature (e.g., see [2]). In [4], the authors suggest the use of the periodogram as a means of estimating a pulse train periodicity by maximising (14) as a function of trial angular frequency ω.

It is also noteworthy that the jitter distribution assumed in [4] corresponds to the *von Mises* distribution, although the authors do not state this fact. For a unimodal von Mises distribution symmetric about the circular mean μ_ϕ, the distribution is described by

$$f(\phi) = \frac{\exp[\kappa\cos(\phi-\mu_\phi)]}{2\pi I_0(\kappa)}, \qquad 0 \le \phi < 2\pi, \qquad \kappa \ge 0, \tag{15}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, and the parameter κ is termed the *concentration parameter*. The von Mises distribution features prominently in circular statistics and based on some, though not all of the distribution's properties, it is sometimes considered to be the circular statistics analogue of the Gaussian distribution of linear statistics. For large κ the distribution can be closely approximated by the wrapped normal distribution [7]. In [1] the authors show that there is only one circular distribution, namely the von Mises distribution, for which the estimator of Equation (7) corresponds to the maximum likelihood estimator of the mean angle μ_ϕ.

4. PARAMETER ESTIMATOR PROPERTIES

In this section we examine several large sample statistical properties associated with the proposed estimators for TOA phase, t_ϕ, and jitter variance, σ_ϵ^2.

4.1. Convergence of Estimates

We begin by assuming that the phase data vector, $\boldsymbol{\phi} = [\phi_0 \;\; \phi_1 \;\; \cdots \;\; \phi_{N-1}]^T$, is available from the measured event TOA sequence folded with the appropriate period T. Let $\gamma_n = \exp(i\phi_n)$ denote a new random variable associated with the nth phase ϕ_n. Given a sequence of N random variables $\{\gamma_0, \; \gamma_1, \; \ldots, \; \gamma_{N-1}\}$, the sample mean, $\bar{\gamma}_N$, is computed using

$$\bar{\gamma}_N = \frac{1}{N} \sum_{n=0}^{N-1} \exp(i\phi_n). \tag{16}$$

From the strong law of large numbers and for i.i.d. random variables $\{\gamma_N\}$, the sample mean, $\bar{\gamma}_N$, converges almost surely to $E\{\gamma_n\}$ [9], i.e., it converges with probability one as $N \to \infty$, where $E\{\gamma_n\} = E\{\exp(i\phi_n)\}$ is the characteristic function (c.f.) of the random variable ϕ_n. If we denote the c.f. of ϕ_n by $\varphi_\phi(1)$[1], we obtain that for the wrapped normal distribution with circular mean, μ_ϕ, and variance, σ_ϕ^2,

$$\varphi_\phi(1) = \exp(i\mu_\phi) \exp\left(-\frac{1}{2}\sigma_\phi^2\right). \tag{17}$$

Hence,

$$\bar{\gamma}_N \stackrel{a.s.}{\to} \exp(i\mu_\phi) \exp\left(-\frac{1}{2}\sigma_\phi^2\right), \tag{18}$$

where the notation $\stackrel{a.s.}{\to}$ is used to denote "converges almost surely to." Also, given a continuous function $g(\cdot)$, we have the following result

$$g(\bar{\gamma}_N) \stackrel{a.s.}{\to} g(E\{\gamma_n\}), \tag{19}$$

since $\bar{\gamma}_N \stackrel{a.s.}{\to} E\{\gamma_n\}$. We therefore obtain that for the estimate of circular mean given by Equation (7)

$$\begin{aligned} \hat{\mu}_\phi &= \arg(\bar{\gamma}_N) \stackrel{a.s.}{\to} \arg(\varphi_\phi(1)), \\ &\stackrel{a.s.}{\to} \mu_\phi. \end{aligned} \tag{20}$$

It then follows from (12) that $\hat{t}_\phi \stackrel{a.s.}{\to} t_\phi$.

Similarly, from Equation (10) we have that for the mean vector length

$$\begin{aligned} r &= |\bar{\gamma}_N| \stackrel{a.s.}{\to} |\varphi_\phi(1)|, \\ &\stackrel{a.s.}{\to} \exp\left(-\frac{\sigma_\phi^2}{2}\right). \end{aligned} \tag{21}$$

Hence, from (8) we have that for the estimate of circular variance

$$\widehat{\sigma_\phi^2} \stackrel{a.s.}{\to} -2\ln\left[\exp\left(-\frac{\sigma_\phi^2}{2}\right)\right] \stackrel{a.s.}{\to} \sigma_\phi^2, \tag{22}$$

[1]Our choice of notation is based on the fact that the c.f. of a Gaussian random variable, X say, with distribution $\mathcal{N}(\mu, \sigma^2)$, is $\varphi_X(s) = \exp(i\mu s)\exp(-\sigma^2 s^2/2)$, and we evaluate this for $s = 1$.

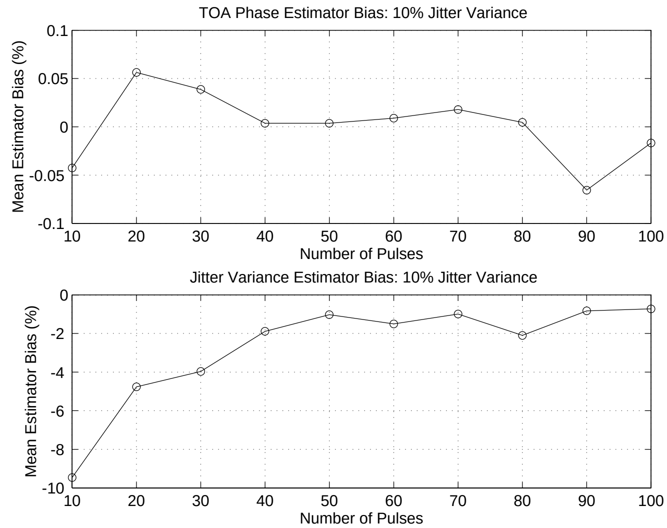

Figure 1. Plots indicating the sample bias in the pulse train TOA phase (upper plot) and jitter variance (lower plot) estimates. 10^3 Monte Carlo realisations were generated for each pulse train set comprising a fixed number of pulses.

and using Equation (13) we obtain the result that $\widehat{\sigma_\epsilon^2} \stackrel{a.s.}{\to} \sigma_\epsilon^2$. The estimators $\hat{t}_\phi$ and $\widehat{\sigma_\epsilon^2}$ are then said to be *statistically consistent* estimators since consistency requires a convergence in probability of an estimate to its true value.

4.2. Unbiasedness and Efficiency of Estimators

Approximate expressions for the mean and variance of the proposed circular statistics based estimators for TOA phase and jitter variance may be obtained using a first-order Taylor series expansion for the estimator expressions. The validity of this method generally improves as the number of data samples increases [6], and with this approach one may show that the TOA phase and jitter variance estimators are asymptotically unbiased and have variances that asymptotically achieve the appropriate Cramér-Rao lower bounds (see below). To ascertain how large N must be for the asymptotic results to become valid we resort to Monte Carlo simulations. The mean percentage bias for each estimator is shown in Figure 1 for pulse train TOA data sets without missing observations. For each pulse train set comprising a fixed number of TOA events, ranging from 10–100 TOAs, 10^3 Monte Carlo realisations were generated. The sample bias was obtained for each pulse train set by computing the difference between the true parameter value and the mean of the estimated parameter. The true parameter values were $t_\phi = 0.5T$ with σ_ϵ^2 corresponding to 10% jitter, where $T = 1.0\,\text{ms}$. From the Taylor series analysis, we would expect both of the circular statistics based estimators to yield asymptotically unbiased estimates. However, inspection of Figure 1 reveals the bias in $\hat{t}_\phi$ to be negligible even for small sample sizes. For $\widehat{\sigma_\epsilon^2}$ on the other hand, a negative bias is evident unless N is moderately large and a minimum number of pulses, 100 say, is required before the bias in the estimate is acceptably small for this level of percentage jitter.

4.2.1. Cramér-Rao Lower Bounds

The Cramér-Rao lower bound (CRLB) provides a theoretical lower limit on the variance that an unbiased parameter estimator can achieve. In quantifying the performance of the circular statistics based estimators we will derive CRLBs with the assumption that the $\{k_n\}$ of the linear model defined by Equation (4) are *available*, although this knowledge is not actually required by the circular data estimators. We will assume that N TOA measurements are available as described by Equation (4). Given an event TOA data vector $\boldsymbol{t} = [t_0 \; t_1 \; \ldots \; t_{N-1}]^T$, and unknown parameter vector $\boldsymbol{\theta} = [t_\phi \; \sigma_\epsilon^2]^T$, the probability density function of the data as a function of $\boldsymbol{\theta}$, or the likelihood function, $p(\boldsymbol{t};\boldsymbol{\theta})$, is

$$p(\boldsymbol{t};\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma_\epsilon^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum_{n=0}^{N-1}(t_n - t_\phi - k_n T)^2\right]. \quad (23)$$

The CRLB for the estimates of the two elements of the vector $\boldsymbol{\theta}$ can be derived from the 2×2 Fisher Information matrix $\boldsymbol{I}(\boldsymbol{\theta})$. For the ith element of $\boldsymbol{\theta}$, the CRLB is computed from $Var(\hat{\boldsymbol{\theta}}_i) \geq [\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}$ [6], where $[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}$ denotes the ith diagonal element of $\boldsymbol{I}^{-1}(\boldsymbol{\theta})$, and where

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{bmatrix} -E\left\{\frac{\partial^2 \ln p(\boldsymbol{t};\boldsymbol{\theta})}{\partial t_\phi^2}\right\} & -E\left\{\frac{\partial^2 \ln p(\boldsymbol{t};\boldsymbol{\theta})}{\partial t_\phi \partial \sigma_\epsilon^2}\right\} \\ -E\left\{\frac{\partial^2 \ln p(\boldsymbol{t};\boldsymbol{\theta})}{\partial \sigma_\epsilon^2 \partial t_\phi}\right\} & -E\left\{\frac{\partial^2 \ln p(\boldsymbol{t};\boldsymbol{\theta})}{\partial {\sigma_\epsilon^2}^2}\right\} \end{bmatrix}. \quad (24)$$

Computing $\boldsymbol{I}^{-1}(\boldsymbol{\theta})$ yields the following estimator performance bounds for t_ϕ and σ_ϵ^2

$$Var\{\hat{t}_\phi\} \geq \frac{\sigma_\epsilon^2}{N} \quad \text{and} \quad Var\{\widehat{\sigma_\epsilon^2}\} \geq \frac{2\sigma_\epsilon^4}{N}. \quad (25)$$

4.2.2. MSE Simulation Results

We complete the performance assessment of the proposed estimators by using Monte Carlo simulations to compare the mean square error (MSE) results obtained with each estimator against the CRLBs derived above. The simulation results will also provide some measure of how large the pulse train data sets must be for the previously described asymptotic variance results to hold true. The MSE performance of the TOA phase and jitter variance estimators of Equations (12) and (13) respectively, is shown in Figure 2 as a function of percentage jitter variance using 500 Monte Carlo realisations for each jitter value. The plots correspond to pulse trains with an average of 20% missing pulses removed in accordance with (3), and original TOA records comprising 50 events, with $T = 1\,\text{ms}$, and $t_\phi = 0.75\,\text{ms}$. The value of percentage jitter used for each set of Monte Carlo realisations was varied from 1–50%. The estimates for TOA phase and jitter variance can be expected to meet the CRLBs for large data sets and therefore both estimators are said to be *asymptotically statistically efficient*. From the Monte Carlo simulation results of Figure 2 the estimators achieve the CRLBs for a sample size that is quite moderate. The bias in the estimate of jitter variance is not evident as the bias squared component of MSE is considerably smaller than the variance of the parameter estimate.

5. CONCLUDING REMARKS

We have demonstrated the utility of circular statistics based estimators for a point process signal with missing observations. This work was motivated by TOA based pulse train signal analysis where one is interested in performing signal reconstruction, classification and identification for electronic defence purposes. The circular statistics based estimators for TOA phase and jitter variance perform optimally despite a pulse drop out of at least 20%, and are a natural extension to several existing pulse train detection and separation algorithms including the event TOA folding algorithm and the periodogram.

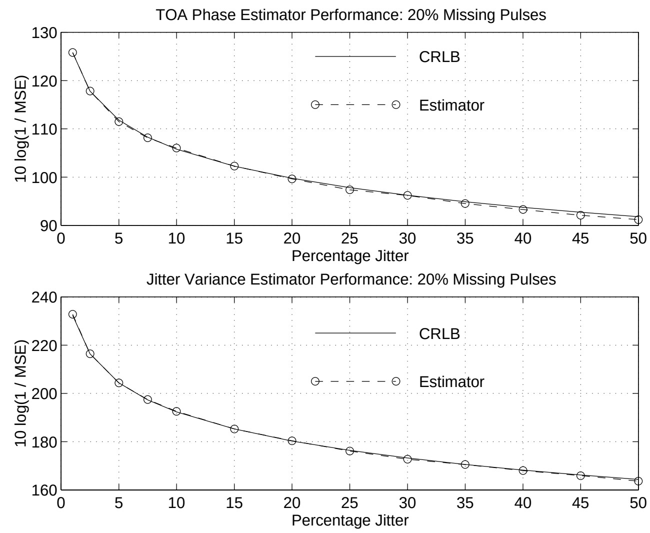

Figure 2. MSE performance of the estimators for pulse train TOA phase (upper plot) and jitter variance (lower plot) for incomplete data sets corresponding to an average of 20% missing pulses from an original 50 TOA events. Results from 500 Monte Carlo realisations are indicated by the broken lines and circles, and compared to the CRLBs indicated by the solid lines.

6. ACKNOWLEDGEMENTS

S.D. Elton would like to thank B.G. Quinn for helpful discussions and to acknowledge funding provided by the Cooperative Research Centre for Robust and Adaptive Systems under the Cooperative Research Centres Programme of the Australian Commonwealth Government.

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