ANALYSIS OF AN LMS ADAPTIVE FEEDFORWARD CONTROLLER FOR PERIODIC DISTURBANCE REJECTION: NON-WIENER SOLUTIONS FOR THE LMS ALGORITHM WITH A NOISY REFERENCE-REVISITED

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ABSTRACT

LMS adaptive cancellation has been found to be effective in various applications of active noise control of periodic disturbances. A deterministic periodic waveform can be used for the reference when the period of the disturbance is known *a priori*. However, the algorithm behavior is determined by so-called Non-Wiener solutions. This paper presents a new vector subspace model for simplifying the analysis of the Non-Wiener behavior. The LMS weights are modelled as a deterministic time-varying mean plus a zero-mean fluctuating part. Each weight component is analyzed separately with the subspace model.

I. INTRODUCTION

Adaptive cancellation of periodic disturbances is required in several applications of active noise cancellation (i.e. noise from rotating and reciprocating machines). Effective control of the periodic disturbances can be achieved using adaptive feedforward cancellation [1-3]. Fig 1 displays the block diagram of a feedforward control system using an adaptive filter W(n) to reject periodic disturbances. This and other similar structures have been recently proposed [1,2] without a detailed analysis of the adaptive filter behavior.

The LMS algorithm is often used for the adaptation because it is easy to implement and is relatively well-understood for stochastic inputs. A deterministic periodic waveform can be used for reference when the period of the disturbance is known *a priori*, However, the behavior of the algorithm is significantly different than when the reference is stochastic. Non-Wiener [4] solutions of the LMS algorithm occur and are due to the non-stationarity of the sinusoidal inputs.

Shensa [5] was the first to analyze the Non-Wiener solutions of the LMS algorithm with a noisy reference. However, his results do not extend to multiple sinusoidal references and to active noise cancellation problems with filters in the cancellation loop (Figure 1 is a Filtered-X-LMS [6,7] structure). [8] presents a simplified orthogonal subspace decomposition method for the noiseless reference case. This approach is extended here to the noisy reference case and to noisy multiple sinusoidal references.

II. PRELIMINARIES

Let $Y(n) = [y(n), y(n-1), \ldots\ldots y(n-N+1)]^T$ denote the *N*-dimensional reference input vector and W(n) denote the *N*-dimensional tap weight vector. W(n) is adjusted by the LMS algorithm according to the recursion

$$W(n+1) = W(n) + \mu\, e(n) Y(n) \qquad (1)$$

$$e(n) = c(n) - W^T(n) Y(n) \qquad (2)$$

c(n) is an external desired signal and

$$W^T(n) Y(n) = \sum_{i=1}^{N} w_i(n) y(n-i+1). \qquad (3)$$

The reference input consists of harmonically related deterministic sinusoids in additive white noise. Thus,

$$y(n) = \sum_{m=1}^{N/2} x_m(n) + v(n) \qquad (4)$$

where

$$x_m(n) = \sqrt{\frac{P_{sm}}{2}}\left[e^{j\pi mn/N} + e^{-j\pi mn/N}\right], \qquad (5)$$

for m=1,2.. N/2. $v(n)$ is white gaussian sequence with $E[v(n)v(m)] = P_n\delta(n-m)$. Each of the sinusoidal components has power P_{s_m}. Y(n) can be written as a signal vector X(n) and a noise vector V(n),

$$Y(n) = \sum_{m=1}^{N/2} X_m(n) + V(n) \quad , \tag{6}$$

$$X_m(n) = \sqrt{\frac{P_{s_m}}{2}}\left[e^{j\pi mn/N} d_m + e^{-j\pi mn/N} d_m^*\right] \tag{7}$$

$$d_m = \left[1, e^{-j\pi m/N}, \ldots\ldots, e^{-j(n-1)\pi m/N}\right]^T, \tag{8}$$

and

$$V(n) = [v(n), v(n-1), \ldots\ldots, v(n-N+1)]^T. \tag{9}$$

Note that both the complex exponentials and d_m and d_m^* are orthogonal. Using (7)-(9), the correlation matrix of Y(n) is

$$E\left[Y(n)Y^T(n)\right] = P_n I + \frac{1}{2}\sum_{m=1}^{N/2} P_{s_m} Q_m(n) \quad , \tag{10}$$

$$Q_m(n) = e^{j2\pi mn/N} d_m d_m^T + d_m^* d_m^T + d_m d_m^{*T} + e^{-j2\pi mn/N} d_m^* d_m^{*T} \quad . \tag{11}$$

The desired (primary) waveform is the sum of harmonically related sinusoids with different powers and independent uniformly distributed random phases,

$$c(n) = \sum_{p=1}^{N/2} \sqrt{\frac{P_{L,p}}{2}}\left[e^{j(\pi pn/N + \theta_p)} + e^{-j(\pi pn/N + \theta_p)}\right] \tag{12}$$

Thus, c(n) is wide-sense stationary.

III. ANALYTICAL RESULTS

The analysis proceeds in two steps, based on the equivalent model for the weights shown in Fig. 2. The first step evaluates the output of the mean system $E[W(n)]$ to the deterministic portion of the input (i.e. the mean response of eq.(1)). The second step evaluates the correlation function of the random portion of the output which is due to three terms - 1) the response of the mean weights to the random input, 2) the response of the random system $\beta(n)$ to the deterministic input and 3) the response of the random part $\beta(n)$ to the random input V(n).

A. One Sinusoidal Reference and One Primary Sinusoid

1. Response of the Mean System to the Deterministic Input

Averaging both sides of eq. (1) yields

$$E[W(n+1)] = \left[(1-\mu P_n)I - \left(\frac{\mu P_s}{2}\right)Q_M(n)\right] \times E[W(n)] + \mu c(n)X(n) \tag{13}$$

Here c(n) is one sinusoid with p=M and initially θ_M is set to zero. Eq. (13) is identical in form to the weight recursion for the noise-free case analyzed in [8]. The mean filter output is given by [9]

$$E\left[W^T(n)\right]E[Y(n)] = E\left[W^T(n)\right]X(n) = \frac{NP_s/2}{P_n + NP_s/2}\sqrt{2}\cos\left(\frac{\pi Mn}{N}\right) \tag{14}$$

The portion of c(n) remaining in the error e(n) is $(1+NP_s/2P_n)^{-1}$ c(n).

2. Correlation Function of the Adaptive Filter Output

The correlation function of the adaptive filter output is given by [9]

$$E\left[W^T(n)Y(n)Y^T(m)W(m)\right] =$$

$$\left[\frac{NP_s/2}{(P_n + NP_s/2)}\right]^2 2\cos\left(\frac{\pi Mn}{N}\right)\cos\left(\frac{\pi Mm}{N}\right) + \frac{NP_nP_s/2}{(P_n + NP_s/2)^2}\delta(n-m) +$$

$$\mu P_n^2\left[\frac{2}{2(P_n + NP_s/2)} + \frac{(N-2)}{2P_n}\right]\delta(n-m) + \mu P_n\left(\frac{NP_s/2}{P_n + NP_s/2}\right)[1-\mu(P_n + NP_s/2)]^{|n-m|} \times$$

$$\cos[\pi M(n-m)/N] + \frac{NP_s}{P_n + NP_s/2}P_n \times [1-\mu(P_n + NP_s/2)]^{n-m-1}Z(n-m-1)$$

$$\times\begin{bmatrix}\cos(\pi mM/N)\cos^2(\pi M(n-m)/N) + \\ \sin(\pi mM/N)\sin(\pi M(n-m)/N) \times \\ \cos(\pi M(n-m)/N)\end{bmatrix} \tag{15}$$

where Z(n-m-1) is a shifted unit step function and is zero for n=m. Eq.(15) assumes $n \geq m$, without loss, and takes into account the correlation between the present weight fluctuations and the past values of V. The autocorrelation function of the error is

$$E[e(n)e(m)] = c(n)c(m) - c(n)E\left[W^T(m)Y(m)\right] - c(m)E\left[W^T(n)Y(n)\right] + E\left[W^T(n)Y(n)Y^T(m)W(m)\right] \qquad (16)$$

For comparison with [5-eq.(29)], c(n) must be modelled by a unit power sinusoid with uniformly distributed phase. Thus,

$$c(n) = \sqrt{2}\cos\left[(\pi pn / N) + \theta_p\right], \qquad (17)$$

$$E[c(n)c(m)] = \cos[\pi p(n-m)/N] \qquad (18)$$

Using (15) and (17) in (16) yields

$$E[e(n)e(m)] = \left(1 + \frac{NP_s}{2P_n}\right)^{-2}\cos(\pi M(n-m)/N) + \frac{NP_nP_s/2}{(P_n + NP_s/2)^2}\delta(n-m) + \mu P_n\left(\frac{NP_s/2}{P_n + NP_s/2}\right) \times\left[(1 - \mu(P_n + NP_s/2))\right]^{|n-m|}\cos[\pi M(n-m)/N] + \mu P_n^2\left[\frac{2}{2(P_n + NP_s/2)} + \frac{(N-2)}{2P_n}\right]\delta(n-m) \qquad (19)$$

After some effort, it can be shown that (19) agrees with [5-eq.(29)]. Note that the analysis has verified the structural equivalence shown in Figure 2.

Eq.(19) implies that the power spectrum of the error corresponds to i) a primary signal residual sinusoidal component which is inversely proportional to the square of the SNR, ii) A white noise component which is inversely proportional to the factor $(1+NP_s/2P_n)$. iii) A narrowband component, centered at frequency M/2N, with amplitude proportional to μP_n, except in the neigborhood of frequency M/2N where the amplitude is inversely proportional to the factor $(1+NP_s/2P_n)$, iv) A white noise component whose spectral amplitude is μNP_n.

The LMS algorithm minimizes the error e(n) by cancelling much of the desired signal c(n), but not all. The cancellation of c(n) is displayed by (i). The algorithm pays a price, however, by introducing terms (ii)-(iv) in the spectrum of the error signal.

For most practical purposes, (19) can be approximated by

$$E[e(n)e(m)] \approx \left(1 + \frac{NP_s}{2P_n}\right)^{-2}\cos(\pi M(n-m)/N) + \frac{NP_nP_s/2}{(P_n + NP_s/2)^2}\delta(n-m) \qquad (20)$$

Note that the ratio of error power to primary signal power is $(1+NPs/2Pn)^{-1}$. Hence, the cancellation performance is directly proportional to the SNR of the reference sinusoids. This would suggest that it is important to minimize the noise in the reference.

B. Multiple Sinusoidal References and Multiple Primary Sinusoids

It is shown in [9] that the dominant terms of the autocorrelation function of the e(n) are given by

$$E[e(n)e(m)] = \sum_{p=1}^{N/2}\left(1 + \frac{NP_{s,p}}{2P_n}\right)^{-2} P_{I,p}\cos\left(\frac{\pi p(n-m)}{N}\right) + P_n\sum_{p=1}^{N/2} P_{I,p}\frac{NP_{s,p}/2}{(P_n + NP_{s,p}/2)^2}\delta(n-m) \qquad (21)$$

Eq.(21) shows that both the sinusoidal errors and the background white noise level increase as the number of desired (primary) sinusoids increases. The first increase is to be expected. The second increase is somewhat surprising. Each cancelled primary sinusoid results in a sinusoidal residual plus a white noise component. Both are scaled by the level of the primary sinusoid. Additional primary sinusoids generate sinusoidal residuals plus additional white noise which is uncorrelated with the white noises caused by the other sinusoids. This behavior is due to the orthogonality of the mean components of E[W(n)] [9]. Consider the special case of $P_{I,p} = P_I$ and $P_{s,p} = P_s$ for all p. Then (21) simplifies to

$$E[e(n)e(m)] = \frac{NP_I}{2}\left(1 + \frac{NP_s}{2P_n}\right)^{-2}\cos\left(\frac{\pi p(n-m)}{N}\right) + \frac{NP_nP_I}{2}\frac{NP_s/2}{(P_n + NP_s/2)^2}\delta(n-m) \qquad (22)$$

Compare (20) and (22). The multiple sine wave canceller pays a performance price. Each additional primary sine wave increases the white noise background floor. Notice, however, that the ratio of the residual error power to the primary power is unchanged.

IV. RESULTS AND CONCLUSIONS

This paper has presented a vector space model for studying the Non-Wiener behavior of the LMS algorithm. The model was used to study two cases: 1) one sinusoidal reference in additive white gaussian noise and one desired (primary) sinusoid without noise, 2) multiple sinusoidal references in additive white gaussian noise and multiple desired sinusoids without noise. The analysis results in (1) agree with [5]. The analysis results in (2) are new and provide support for the usefulness of the model. (2) demonstrates that the multiple sinusoid reference canceller can cancel multiple desired sinewaves in the primary. However, the backgound noise floor of the error signal grows in proportion to the number of signals and their relative powers.

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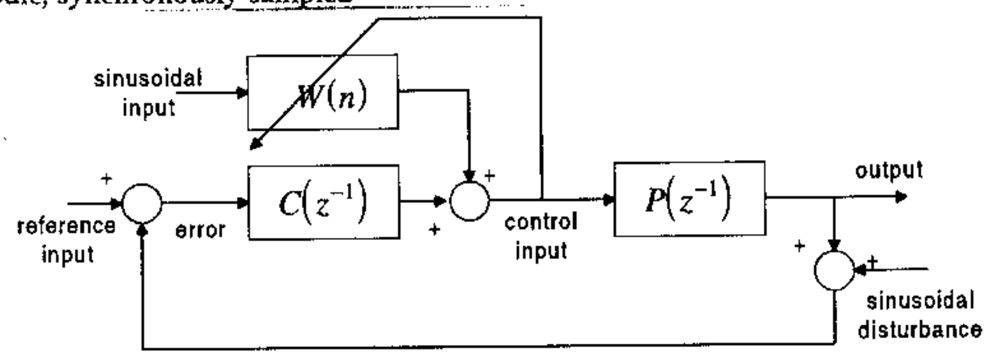

Fig.1 - Block diagram of feedforward control for rejecting periodic disturbances.

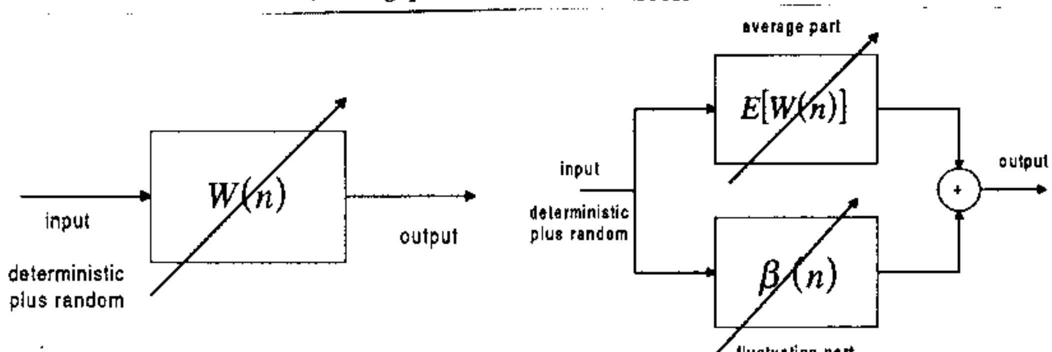

Fig. 2 - Equivalent structures for the adaptive filter weights.