AN ADAPTIVE ESPRIT ALGORITHM BASED ON PERTURBATION OF UNSYMMETRICAL MATRICES

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ABSTRACT

Many subspace updating algorithms based on the eigenvalue decomposition (EVD) of array covariance matrices have been proposed and used in high-resolution array processing algorithms in recent years. In some applications (i.e. ESPRIT algorithms), however, the EVD of an unsymmetrical matrix is also needed. In this paper, an EVD updating approach for an unsymmetrical matrix is presented based on its first-order perturbation analysis. By jointly using this approach and a subspace updating method in an ESPRIT algorithm, a completely adaptive ESPRIT algorithm is obtained. The evaluation of the complexity and the performance of this algorithm is given in the paper.

1 INTRODUCTION

In recent years, much attention has been drawn towards the application of the high-resolution frequency and direction-of-arrival (DOA) estimation techniques to nonstationary environments. Several adaptive eigendecomposition algorithms, e.g. [1]-[3] and references therein, have been developed to make the subspace-based approaches such as MUSIC applicable for tracking time-varying DOAs.

ESPRIT [4]-[5] is known to be more computationally efficient in DOA estimation than MUSIC-type algorithms. It usually consists of two steps. In the case of uniform linear array with L elements and a wavefield consisting of M sources, the first step is to estimate the $L\times M$ matrix of eigenvectors associated to the signal subspace, just like in MUSIC-type algorithms. In the second step, this information is used to estimate the DOAs. In TLS-ESPRIT [4], this involves one $2M\times 2M$ eigendecomposition and one $M\times M$ generalized eigendecomposition, while in GEESE-ESPRIT [5], this involve one $L\times M$ generalized eigendecomposition. If M is not so small compared with L, for example $M = L/2$, GEESE-ESPRIT will be more computationally efficient than TLS-ESPRIT.

Recently, an adaptive TLS-ESPRIT algorithm [6] has been proposed for tracking time-varying signals. In this algorithm, the URV decomposition is used for both the $L\times M$ signal subspace decomposition and the $2M\times 2M$ eigendecomposition while no special consideration is given to the $M\times M$ generalized eigendecomposition. In this paper, we first develop an approximate eigendecomposition for unsymmetrical matrices based on a first-order perturbation approach and then apply it to GEESE-ESPRIT to produce a new completely adaptive ESPRIT algorithm.

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2 FIRST-ORDER PERTURBATION OF UNSYMMETRICAL MATRICES

Let $H_0 \in C^{M\times M}$ be a matrix which is modified by a small perturbation. The perturbed matrix can be expressed as

$$H(\varepsilon) = H_0 + \varepsilon H_1 \qquad (1)$$

where H_1 is a perturbation matrix of H_0 and is a small real factor.

Assume $H(\varepsilon)$ has distinct eigenvalues. Then there exist linearly independent vectors $\{\boldsymbol{q}_1(\varepsilon), \boldsymbol{q}_2(\varepsilon), \ldots, \boldsymbol{q}_M(\varepsilon)\}$ and $\{\boldsymbol{p}_1(\varepsilon), \boldsymbol{p}_2(\varepsilon), \ldots, \boldsymbol{p}_M(\varepsilon)\}$ in $C^{M\times 1}$ and scalars $\{\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_M(\varepsilon)\}$ such that

$$H(\varepsilon)\boldsymbol{q}_i(\varepsilon) = \lambda_i(\varepsilon)\boldsymbol{q}_i(\varepsilon) \qquad (2)$$

$$\boldsymbol{p}_i^H(\varepsilon)H(\varepsilon) = \lambda_i(\varepsilon)\boldsymbol{p}_i^H(\varepsilon) \qquad (3)$$

$$\boldsymbol{p}_i^H(\varepsilon)\boldsymbol{q}_j(\varepsilon) = \delta_{ij} \qquad (4)$$

where δ_{ij} is the Kronecker delta. The vectors $\boldsymbol{q}_i(\varepsilon)$ and $\boldsymbol{p}_i(\varepsilon)$ are known as right and left eigenvectors of $H(\varepsilon)$, respectively, and $\lambda_i(\varepsilon)$ are the corresponding eigenvalues.

Note that when $\varepsilon = 0$, $\{\boldsymbol{q}_i(0), \boldsymbol{p}_i(0), \lambda_i(0)\}$ are the right eigenvectors, left eigenvectors and corresponding eigenvalues of H_0, respectively. For simplicity, these quantities are denoted as

$$\boldsymbol{q}_{i0} = \boldsymbol{q}_i(0), \quad \boldsymbol{p}_{i0} = \boldsymbol{p}_i(0), \quad \lambda_{i0} = \lambda_i(0) \qquad (5)$$

In many applications, $H(\varepsilon)$ represents an updating form of H_0. The so-called adaptive eigendecomposition of $H(\varepsilon)$ is to find $\{\boldsymbol{q}_i(\varepsilon), \boldsymbol{p}_i(\varepsilon), \lambda_i(\varepsilon)\}$ with the knowledge of old quantities $\{\boldsymbol{q}_{i0}, \boldsymbol{p}_{i0}, \lambda_{i0}\}$ and the new data which is contained in the perturbation matrix H_1.

Since $H(\varepsilon)$ has distinct eigenvalues, based on the perturbation theorem of matrices ([7], pp.66), $\lambda_i(\varepsilon)$, $\boldsymbol{q}_i(\varepsilon)$ and $\boldsymbol{p}_i(\varepsilon)$ can be expressed in the following power series, which are all convergent in a neighborhood of $\varepsilon = 0$

$$\lambda_i(\varepsilon) = \lambda_{i0} + \lambda_{i1}\varepsilon + \lambda_{i2}\varepsilon^2 + \ldots \qquad (6)$$

$$\boldsymbol{q}_i(\varepsilon) = \boldsymbol{q}_{i0} + \boldsymbol{q}_{i1}\varepsilon + \boldsymbol{q}_{i2}\varepsilon^2 + \ldots \qquad (7)$$

$$\boldsymbol{p}_i(\varepsilon) = \boldsymbol{p}_{i0} + \boldsymbol{p}_{i1}\varepsilon + \boldsymbol{p}_{i2}\varepsilon^2 + \ldots \qquad (8)$$

where λ_{ij}, $\boldsymbol{q}_{ij}$ and $\boldsymbol{p}_{ij}$ correspond to the jth-order derivatives of $\lambda_i(\varepsilon)$, $\boldsymbol{q}_i(\varepsilon)$ and $\boldsymbol{p}_i(\varepsilon)$ at $\varepsilon = 0$, respectively.

In practice, perturbation series (6)-(8) are truncated at a certain order to yield tractable approximations. This approach is justified on the basis that the perturbation series are convergent in some neighborhood of $\varepsilon = 0$. Hence, provided ε is sufficiently small, a low-order approximation can be used to evaluate $\lambda_i(\varepsilon)$, $\boldsymbol{q}_i(\varepsilon)$ and $\boldsymbol{p}_i(\varepsilon)$ with good approximation. This philosophy has been used in [3] to obtain an adaptive eigendecomposition of data covariance matrices. In this work, a first-order approximation of (6)-(8) is used to update eigenvectors and eigenvalues of the unsymmetrical matrix $H(\varepsilon)$.

The values of λ_{i1}, $\boldsymbol{q}_{i1}$ and $\boldsymbol{p}_{i1}$ in (6)-(8) can be obtained by substituting (1) into (2)-(4), differentiating the resulting expressions with respect to ε and setting $\varepsilon = 0$, which results in

$$H_0\boldsymbol{q}_{i1} + H_1\boldsymbol{q}_{i0} = \lambda_{i0}\boldsymbol{q}_{i1} + \lambda_{i1}\boldsymbol{q}_{i0} \tag{9}$$

$$\boldsymbol{p}_{i1}^H H_0 + \boldsymbol{p}_{i0}^H H_1 = \lambda_{i0}\boldsymbol{p}_{i1}^H + \lambda_{i1}\boldsymbol{p}_{i0}^H \tag{10}$$

$$\boldsymbol{p}_{j0}^H\boldsymbol{q}_{i1} + \boldsymbol{p}_{j1}^H\boldsymbol{q}_{i0} = 0 \tag{11}$$

Solving the above equations and using the equalities in (2)-(4) for $\varepsilon = 0$, we have

$$\lambda_{i1} = \boldsymbol{p}_{i0}^H H_1 \boldsymbol{q}_{i0} \tag{12}$$

$$\boldsymbol{q}_{i1} = \sum_{j=1, j\neq i}^{M} \mu_{ij}\boldsymbol{q}_{j0} \tag{13}$$

$$\boldsymbol{p}_{i1} = -\sum_{j=1, j\neq i}^{M} \mu_{ji}{}^*\boldsymbol{p}_{j0} \tag{14}$$

where

$$\mu_{ij} = \frac{\boldsymbol{p}_{j0}^H H_1 \boldsymbol{q}_{i0}}{\lambda_{i0} - \lambda_{j0}} \tag{15}$$

Substituting (12)-(14) into (6)-(8) respectively and neglecting the higher order terms in ε^2, we obtain the following estimates of the eigenvalues and eigenvectors of the perturbed unsymmetrical matrix $H(\varepsilon)$

$$\lambda_i(\varepsilon) \approx \lambda_{i0} + (\boldsymbol{p}_{i0}^H H_1 \boldsymbol{q}_{i0})\,\varepsilon \tag{16}$$

$$\boldsymbol{q}_i(\varepsilon) \approx \boldsymbol{q}_{i0} + \varepsilon \sum_{j=1, j\neq i}^{M} \mu_{ij}\boldsymbol{q}_{j0} \tag{17}$$

$$\boldsymbol{p}_i(\varepsilon) \approx \boldsymbol{p}_{i0} - \varepsilon \sum_{j=1, j\neq i}^{M} \mu_{ji}{}^*\boldsymbol{p}_{j0} \tag{18}$$

If $H(\varepsilon)$ is Hermitian, it results $\boldsymbol{p}_i(\varepsilon) = \boldsymbol{q}_i(\varepsilon)$ and the above equations (16)-(18) for EVD updating will be identical to that used in [3]. In the sequel, this approach will be applied to array signal processing to obtain an efficient updating algorithm for ESPRIT.

3 EIGENSTRUCTURE METHODS FOR DOA ESTIMATION

3.1 ESPRIT

Denote $\boldsymbol{x}(t) \in C^{L\times 1}$ as the received complex data vector with a uniform linear array (ULA) of L elements and let $R_x = E[\boldsymbol{x}(t)\boldsymbol{x}^H(t)]$ be the array covariance matrix. The so-called eigenstructue methods for DOA estimation are based on the eigendecomposition of R_x, and more specifically, on the estimation of the signal subspace which is spanned by the signal eigenvector matrix V

$$V = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M] \tag{19}$$

where M is the number of signal sources and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_M$ are the M signal eigenvectors of R_x.

In the ULA case, further processing based on the signal subspace leads to different DOA estimation algorithms. Among them, the ESPRIT is more efficient compared to the MUSIC and Root-MUSIC algorithms. In the ESPRIT, the signal eigenvector matrix V is partitioned into two $(L-1)\times M$ overlapping matrices F and G

$$F = [I_{L-1}\ \ \boldsymbol{0}]\,V \tag{20}$$

$$G = [\boldsymbol{0}\ \ I_{L-1}]\,V \tag{21}$$

where I_{L-1} is the $(L-1)\times(L-1)$ identify matrix and $\boldsymbol{0}$ is the $(L$-1)-dimensional zero vector. The DOA estimates are extracted from further eigen-analysis of the matrix pair $\{F, G\}$. For example, in GEESE-ESPRIT [5], only non-zero generalized eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_M$ of $\{F, G\}$ are needed to be calculated. The values for DOAs are then easily solved from $\gamma_1, \gamma_2, \ldots, \gamma_M$ by using the phase relation between array elements.

Let $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_M$ be the generalized eigenvectors of $\{F, G\}$ corresponding to the eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_M$, we have

$$F\boldsymbol{q}_m = \gamma_m G\boldsymbol{q}_m \tag{22}$$

Premultipling (22) by G^H yields

$$\bar{F}\boldsymbol{q}_m = \gamma_m \bar{G}\boldsymbol{q}_m \tag{23}$$

where

$$\bar{G} = G^H G \tag{24}$$

$$\bar{F} = G^H F \tag{25}$$

Assume that $\bar{G}$ is invertible, then (23) can be rewritten as:

$$H\boldsymbol{q}_m = \gamma_m \boldsymbol{q}_m \tag{26}$$

where

$$H = \bar{G}^{-1}\bar{F} \tag{27}$$

Thus, the generalized eigenvalues of $\{F, G\}$ can be solved through the eigendecomposition of the matrix H (27). Note that the matrix H is generally unsymmetrical. Accordingly, most of the recently proposed algorithms for eigendecomposition updating are not applicable to solve this problem.

3.2 Adaptive subspace estimation based on first-order perturbation.

In nonstationary signal fields, it is convenient to estimate the array covariance matrix R_x recursively. The commonly used exponential windowing recursive estimate for R_x can be expressed as:

$$R_x(t) = R_x(t-1) + \varepsilon[\boldsymbol{x}(t)\boldsymbol{x}^H(t) - R_x(t-1)] \tag{28}$$

where ε is a small scalar with $0 \le \varepsilon \le 1$. For ε sufficiently small, the modification term $\varepsilon[\boldsymbol{x}(t)\boldsymbol{x}^H(t) - R_x(t-1)]$ in (28) can be interpreted as a small perturbation of $R_x(t$-1). Based on this observation and first-order perturbation analysis of matrices, a new kind of method for efficiently updating the eigendecomposition of $R_x(t)$ is presented in [3]. In this method, the signal eigenvalues and eigenvectors of the matrix $R_x(t)$ are estimated as

$$\lambda_m(t) = \lambda_m(t-1) + \varepsilon\lambda_{m1} \tag{29}$$

$$\boldsymbol{v}_m(t) = \boldsymbol{v}_m(t-1) + \varepsilon\boldsymbol{v}_{m1} \tag{30}$$

where λ_{m1} and $\boldsymbol{v}_{m1}$ (m = 1,..., M) can be regarded, respectively, as the perturbation terms added to eigenvalues and eigenvectors in the previous step. In matrix form, (30) can be expressed as

$$V(t) = [\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \ldots, \boldsymbol{v}_M(t)] \tag{31}$$

$$= V_0 + \varepsilon V_1 \tag{32}$$

where

$$V_0 = [\boldsymbol{v}_1(t-1), \boldsymbol{v}_2(t-1), \ldots, \boldsymbol{v}_M(t-1)] \tag{33}$$

$$V_1 = [\boldsymbol{v}_{11}, \boldsymbol{v}_{21}, \ldots, \boldsymbol{v}_{M1}] \tag{34}$$

The orthonormality of $V(t)$ is also maintained to the first-order approximation, i. e.,

$$V^H(t)V(t) = I \tag{35}$$

4 APPLICATION OF ADAPTIVE EVD OF UNSYMMETRICAL MATRICES TO ESPRIT

In this section, a first-order perturbation expression for H(t) (27) is derived based on the expression of $V(t)$ (32). The eigendecomposition method described in section 2 is then applied to $H(t)$ to update its eigenvalues.

Substituting $V(t)$ in (32) into (20) and (21), we obtain the two $(L\text{-}1)\times M$ sub-matrices $F(t)$ and $G(t)$, which can be expressed as

$$F(t) = F_0 + \varepsilon F_1 \tag{36}$$

$$G(t) = G_0 + \varepsilon G_1 \tag{37}$$

where $F_0 = F(t\text{-}1)$ and $G_0 = G(t\text{-}1)$ are the corresponding sub-matrices of V_0, and F_1 and G_1 are the corresponding sub-matrices of V_1.

Let $\boldsymbol{w}(t) \in C^{M\times 1}$ be the first column of $V^H(t)$. Similarly, it can be expressed as:

$$\boldsymbol{w}(t) = \boldsymbol{w}_0 + \varepsilon \boldsymbol{w}_1 \tag{38}$$

The orthonormality equation (35) of the matrix $V(t)$ can be then expressed in its sub-matrix form as

$$G^H(t)G(t) + \boldsymbol{w}(t)\boldsymbol{w}^H(t) = I \tag{39}$$

Based on the early discussion, we need to express

$$H(t) = \bar{G}(t)^{-1}\bar{F}(t) \tag{40}$$

in the first order approximation form

$$H(t) = H_0 + \varepsilon H_1 \tag{41}$$

where $\bar{G}(t)$ and $\bar{F}(t)$ are defined in (24) and (25), respectively, with the corresponding time variables, and $H_0 = H(t\text{-}1)$. After some derivations, the perturbation term H_1 can be found as

$$H_1 = \bar{G}_0^{-1}(G_1^H F_0 + G_0^H F_1) + \tilde{G}\bar{F}_0 \tag{42}$$

where $\bar{G}_0 = G_0^H G_0$, $\bar{F}_0 = G_0^H F_0$ and

$$\tilde{G} = \alpha_0 \boldsymbol{w}_0 \boldsymbol{w}_1^H + (\alpha_0 \boldsymbol{w}_1 + \alpha_1 \boldsymbol{w}_0)\boldsymbol{w}_0^H \tag{43}$$

$$\alpha_0 = [1 - \boldsymbol{w}_0^H \boldsymbol{w}_0]^{-1} \tag{44}$$

$$\alpha_1 = 2\alpha_0^2 \cdot \mathrm{Re}[\boldsymbol{w}_0^H \boldsymbol{w}_1] \tag{45}$$

In (42), the inverse of $\bar{G}_0 = G_0^H G_0$ can be calculated efficiently by applying the matrix inversion lemma to (39) (with $t \to t-1$), which gives

$$\bar{G}_0^{-1} = I + \alpha_0 \boldsymbol{w}_0 \boldsymbol{w}_0^H \tag{46}$$

With H_1 available, the eigendecomposition of $H(t)$ can be updated by using (15)-(18). Note that in this adaptive EVD method only the coefficients μ_{ij} (15) are required to update the eigenvalues and eigenvectors. Thus, a direct calculation of the matrix H_1 as expressed in (42) can be avoided. A complete algorithm for adaptive ESPRIT is summarized in the Table 1. In this algorithm, an adaptive EVD method from [3] is used for the signal subspace updating: $V_m(t) = V_m(t-1) + V_1$, where the perturbation parameter ε is now included in the perturbation matrix V_1.

In particular, if the PC algorithm in [3] is employed, the total complex multiplication operations per iteration for the algorithm in Table 1 is $5LM + 4M^3 + O(M^2)$, where $5LM$ is the complexity required by the PC algorithm for the signal subspace updating. Actually, by investigating the structure in the expression for the signal subspace matrix $V(t)$ obtained from the PC algorithm, a more efficient way of calculating μ_{ij} can be obtained. As a result, the total computational complexity of the adaptive ESPRIT algorithm can be further reduced. Due to space limitations, this results will no be presented here.

5 SIMULATIONS

Computer simulations were used to evaluate the performance of the new adaptive algorithm. A uniform linear array of L = 8 sensors with half-wavelength spacing is used to monitor M = 2 uncorrelated plane waves corrupted by additional white noise. The first example concerns two fix sources located at 12^o and 9^o, with SNR = 20 dB. The perturbation parameter was set to $\varepsilon = 0.02$. Fig.1 shows the results of DOA estimation after averaging 100 independent runs. The results of exact GEESE-ESPRIT are also presented. It can be seen that the estimates obtained with the new adaptive algorithm are very close to that of the exact GEESE-ESPRIT.

Simulations with time-varying DOAs were also considered. Two moving sources are located at

$$20° + 5°\sin(2\pi t/600), \qquad 10° + 5°\sin(2\pi t/300)$$

and the SNR is 20 dB. In this case, the perturbation parameter was set to $\varepsilon = 0.05$ to increase the tracking capability. Fig.2 shows the DOA tracking results by averaging 100 independent runs. The adaptive ESPRIT algorithm performs very well and differs only slightly from the exact GEESE-ESPRIT.

6 CONCLUSIONS

In the ESPRIT algorithm for DOA estimation, the EVD for both Hermitian matrices and unsymmetrical matrices are needed. While many efficient updating algorithms for the former have been presented and could have been used in the ESPRIT, little work has been done for the fast implementation of the latter. In this work, based on the first-order perturbation analysis of unsymmetrical matrices, an approximate solution of EVD updating for unsymmetrical matrices is provided and a completely adaptive GEESE-ESPRIT algorithm is presented. The complexity of this algorithm is $5LM + 4M^3 + O(M^2)$. Simulations showed that it is effective in tracking time-varying DOAs. The technique used here can also be applied to other ESPRIT algorithms, such as TLS-ESPRIT.

TABLE 1. Adaptive ESPRIT algorithm

Using subspace updating algorithms in [3]:

$$V_m(t) = V_m(t-1) + V_1$$

Obtaining:

F_0, G_0, F_1, G_1, $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ from $V_m(t\text{-}1)$ and V_1

Calculating:

$$\alpha_0 = [1 - \boldsymbol{w}_0^H \boldsymbol{w}_0]^{-1}$$
$$\alpha_1 = 2\alpha_0^2 \cdot \mathrm{Re}\,[\boldsymbol{w}_0^H \boldsymbol{w}_1]$$
$$\boldsymbol{b}_{00} = G_0 \boldsymbol{w}_0\,,\ \boldsymbol{b}_{01} = G_1 \boldsymbol{w}_0\,,\ \boldsymbol{b}_{10} = G_0 \boldsymbol{w}_1$$

Available at time t:

EVD of $H(t\text{-}1)$: $\boldsymbol{p}_{m0}$, $\boldsymbol{q}_{m0}$ and γ_{m0}

Solving EVD of $H(t) = H(t-1) + H_1$:

$$Z = G_0^H F_1 + G_1^H F_0$$
$$\boldsymbol{c}_1 = F_1^H \boldsymbol{b}_{00} + F_0^H (\boldsymbol{b}_{01} + \boldsymbol{b}_{10})$$
$$\boldsymbol{c}_2 = F_0^H \boldsymbol{b}_{00}$$

for m = 1 to M:

$$\rho_1 = \boldsymbol{c}_1^H \boldsymbol{q}_{m0}$$
$$\rho_2 = \boldsymbol{c}_2^H \boldsymbol{q}_{m0}$$
$$\boldsymbol{d}_m = Z\boldsymbol{q}_{m0} + (\alpha_0\rho_1 + \alpha_1\rho_2)\,\boldsymbol{w}_0 + (\alpha_0\rho_2)\,\boldsymbol{w}_1$$

end

for m = 1 to M, i = 1 to M, $i \neq m$

$$\mu_{mi} = \boldsymbol{p}_{i0}^H \boldsymbol{d}_m / (\gamma_{m0} - \gamma_{i0})$$

end

for m = 1 to M:

$$\gamma_m(t) = \boldsymbol{p}_{m0}^H \boldsymbol{d}_m$$
$$\boldsymbol{q}_m(t) = \boldsymbol{q}_{m0} + \sum_{i=1,\, i \neq m}^{M} \mu_{mi} \boldsymbol{q}_{i0}$$
$$\boldsymbol{p}_m^H(t) = \boldsymbol{p}_{m0}^H - \sum_{i=1,\, i \neq m}^{M} \mu_{im} \boldsymbol{p}_{i0}^H$$

end

Solving DOAs from $\gamma_m(t)$

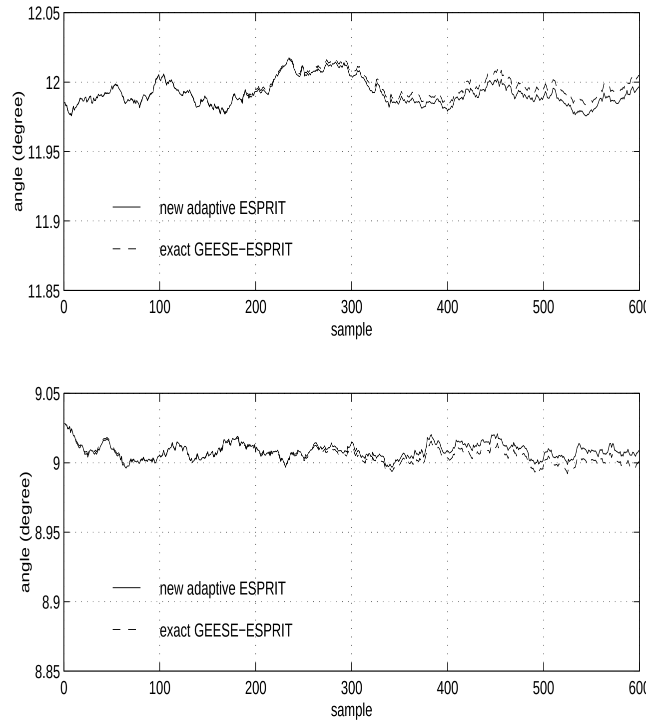

Fig.1 DOA estimates for the new adaptive ESPRIT and the exact GEESE-ESPRIT. Two fixed sources at 12^o and 9^o; SNR = 20 dB; ε = 0.02; 100 independent runs.

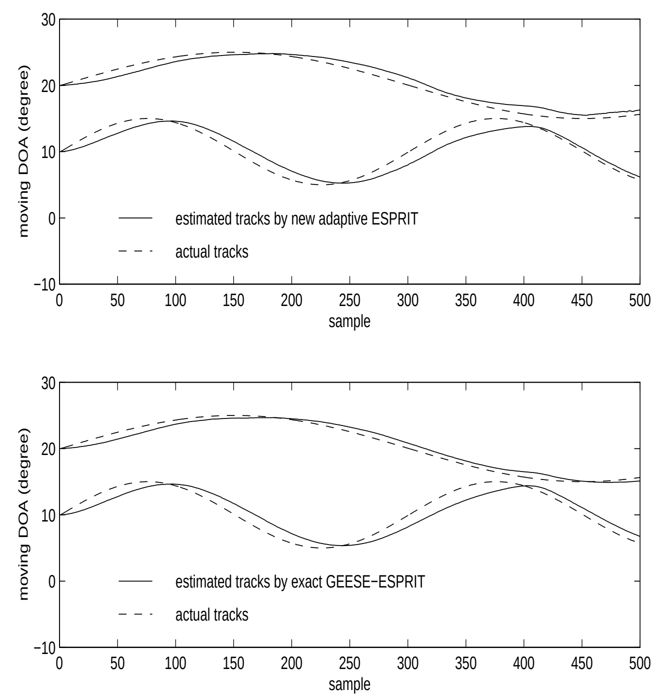

Fig.2. Time-varying DOA estimates for the new adaptive ESPRIT and the GEESE-ESPRIT. Two moving sources at $20° + 5°\sin(2\pi t/600)$ and $10 + 5°\sin(2\pi t/300)$; SNR = 20 dB; ε = 0.05; 100 independent runs.

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